On the level sets of Okamoto's function

R. Dániel Prokaj
Joint with Balázs Bárány

University of North Texas

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Okamoto's function

Okamoto [7] introduced and studied a one-parameter family of self-affine functions

$T_a : [0,1] \to [0,1]$ for $a \in (0,1)$.

Let $a \in (\frac{1}{2}, 1)$, and consider the planar self-affine IFS $\mathcal{F}$.

$$F_1(x,y) = \left(\frac{x}{3}, ay\right)$$

$$F_2(x,y) = \left(\frac{x+1}{3}, (1-2a)y + a\right)$$

$$F_3(x,y) = \left(\frac{x+2}{3}, ay + 1 - a\right)$$

Let $a \in (\frac{1}{2}, 1)$, and consider the planar self-affine IFS $\mathcal{F}$. We often refer to $\mathcal{F}$ as the Okamoto IFS.

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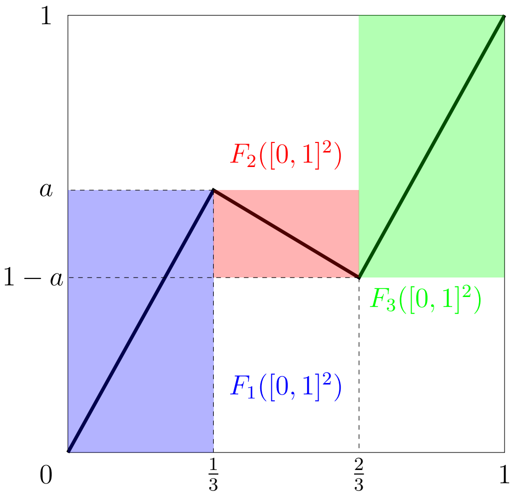

Properties of T_a

$T_{2/3}$ and $T_{5/6}$ were studied by Perkins and Bourbaki repsectively. Okamoto was the first who investigated the whole family.

Theorem (Okamoto 2005 [7])

Let $a_0 \approx 0.5592$ be the unique real root of $54a^3 - 27a^2 = 1$.

(i) $\frac{2}{3} \leqslant a < 1 : T_a$ *is nowhere differentiable,*

(ii) $a_0 < a < \frac{2}{3} : T_a$ *is nondifferentiable at almost every $x \in [0, 1]$, but differentiable at uncountably many points,*

(iii) $0 < a < a_0 : T_a$ *is differentiable almost everywhere, but nondifferentiable at uncountably many points.*

Properties of T_a

Let $\mathcal{D}_\infty(a) = \{x \in [0,1] : T_a'(x) = \pm\infty\}$.

Theorem (Allaart '16 [1])

Let $\hat{a} \approx 0.5598$ *and* $\phi = \frac{\sqrt{5}-1}{2}$.

(i) $\phi \leqslant a < 1 : \mathcal{D}_\infty(a)$ *is empty,*

(ii) $\hat{a} < a < \phi : \mathcal{D}_\infty(a)$ *is countably infinite, containing only rational points,*

(iii) $\frac{1}{2} < a < \hat{a} : \mathcal{D}_\infty(a)$ *is uncountable with* $\dim_{\mathrm{H}} \mathcal{D}_\infty(a) > 0$.

Level sets of T_a

We may gain more insight on the structure of Okamoto's function if look at its level sets.

Theorem (Baker, Bender '23 [3])

Let $a_9 \approx 0.50049$.

(i) $\frac{1}{2} \leqslant a < a_9$: *there exists* $y \in [0,1]$ *for which* $|T_a^{-1}(y)| = 3$,

(ii) *Assume that for every* $y \in [0,1]$, $|T_a^{-1}(y)|$ *is either uncountable or* 1. *Then if* $|T_a^{-1}(y)|$ *is uncountable,* $\exists s > 0$ *such that* $\dim_{\mathrm{H}} T_a^{-1}(y) \geqslant s$.

Dimension theory of the graph

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However, as Allaart pointed it out, his argument was incorrect.

Our main result

For $a \in \left(\frac{1}{2}, 1\right)$ set $s_0(a) = 1 + \frac{\ln(4a-1)}{\ln(3)}$, and write $\mathcal{O}_a = \operatorname{graph}(T_a)$.

Theorem (B. Bárány, P. '24)

There exists an $\mathcal{E} \subset \left(\frac{1}{2}, 1\right)$ *with* $\dim_{\mathrm{H}} \mathcal{E} = 0$ *such that for every* $a \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$ *the following statements hold.*

- $\dim_{\mathrm{H}} \mathcal{O}_a = \dim_{\mathrm{B}} \mathcal{O}_a = \dim_{\mathrm{A}} \mathcal{O}_a = s_0(a)$,
- $\forall y \in [0,1] : \dim_{\mathrm{H}} T_a^{-1}(y) \leqslant s_0(a) - 1$,
- *For* $\mathcal{L}$*-almost every* $y \in [0,1]$:

$$\dim_{\mathrm{H}} T_a^{-1}(y) = s_0(a) - 1.$$

Main tools

Let $\mathcal{S} = \{\mathbf{A}_i\mathbf{x} + \mathbf{t}_i\}_{i=1}^{m}$ be a self-affine iterated function system on the plane, where $\mathbf{A}_i$ is a diagonal matrix for all $i \in \{1, \dots, m\}$.

Let Λ be the attractor of $\mathcal{S}$, and let μ be the projection of a Bernoulli measure on the symbolic space defined by the probability vector $\mathbf{p}$.

Write proj_y for the projection to the y-axis and L_y for the level set of Λ corresponding to y

$$L_y = \{x \in [0,1] : (x,y) \in \Lambda\}.$$

Theorem (Feng-Hu '09 [5])

Let $\mu_y^{\mathrm{proj}_y^{-1}}$ *denote the conditional measure of* μ *with respect to* L_y *for* $y \in [0,1]$. *Let* $h_{\mathbf{p}}$ *be the entropy of* $\mathbf{p}$ *and* $0 < \chi_1 < \chi_2$ *be the Lyapunov exponents of* μ. *If* x *is the dominating direction, then*

(i) $\dim_{\mathrm{H}} \mu_y^{\mathrm{proj}_y^{-1}} + \dim_{\mathrm{H}} (\mathrm{proj}_y)_* \mu = \dim_{\mathrm{H}} \mu$ *for* $(\mathrm{proj}_y)_*\mu$*-almost all* $y \in [0,1]$,

(ii) $\dim_{\mathrm{H}} \mu = \frac{h_{\mathbf{p}}}{\chi_2} + \left(1 - \frac{\chi_1}{\chi_2}\right) \cdot \dim_{\mathrm{H}} (\mathrm{proj}_y)_* \mu$.

To show that the Assouad dimension of the graph of the function is also equal to the affinity dimension, we used the following result.

Theorem (Antilla-Bárány-Käenmäki '23 [2])

If $\mathcal{S}$ satisfies the open set condition, and its dominating direction is x, then

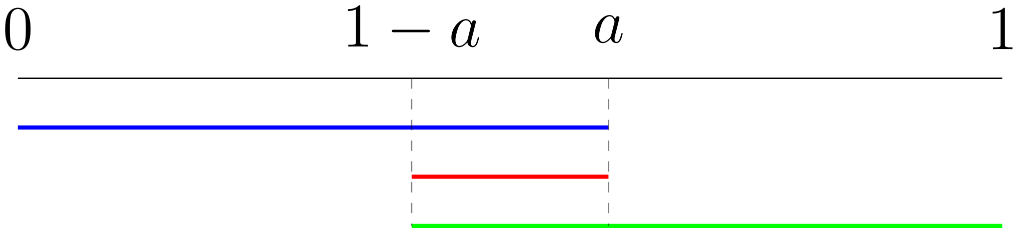
$$\dim_{\mathrm{A}} \Lambda \leqslant \max\{\dim_{\mathrm{H}} \Lambda, 1 + \sup_{y \in [0,1]} \dim_{\mathrm{H}} L_y\}.$$

Why not every $a > 1/2$?

The projection of $\mathcal{O}_a$ to the y-axis contains several overlapping segments. Thus, we can not expect the Hasudorff dimension of the graph to be equal to the affinity dimension for all parameters.

For level sets, it follows from the theorem of Baker-Bender, that for certain parameters we can always find a level set with Hausdorff dimension 0.

Why not every $a > 1/2$?



We can describe the projection to the y-axis with the help of the self-similar IFS

$$\Phi_a = \{f_1(x) = ax,\ f_2(x) = (1-2a)x + a,\ f_3(x) = ax + (1-a)\}.$$

Lemma (B. Bárány, P. '24)

There exists an $\mathcal{E} \subset \left(\frac{1}{2}, 1\right)$ *with* $\dim_{\mathrm{H}} \mathcal{E} = 0$ *such that for every* $a \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$ *the IFS* Φ_a *is exponentially separated.*

Thank you for your attention!

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