A survey on Piecewise Linear Iterated Function Systems on the line

Dániel Prokaj
Joint with K. Simon

Department of Stochastics
Budapest University of Technology and Economics

Interdisciplinary Doctoral Conference, 12 Nov 2021

Introducing piecewise linear iterated function systems

The natural pressure function

Our main results

Introduction

The main object of our investigation is a special family of iterated function systems, defined on the real line.

Definition 1.1

If $\mathcal{F} = \{f_k\}_{k=1}^{m}$ is a finite list of strict contractions on $\mathbb{R}$, then we call $\mathcal{F}$ an iterated function system (IFS).

Each Iterated function system $\mathcal{F}$ uniquely defines an invariant set that we refer to as the attractor of $\mathcal{F}$.

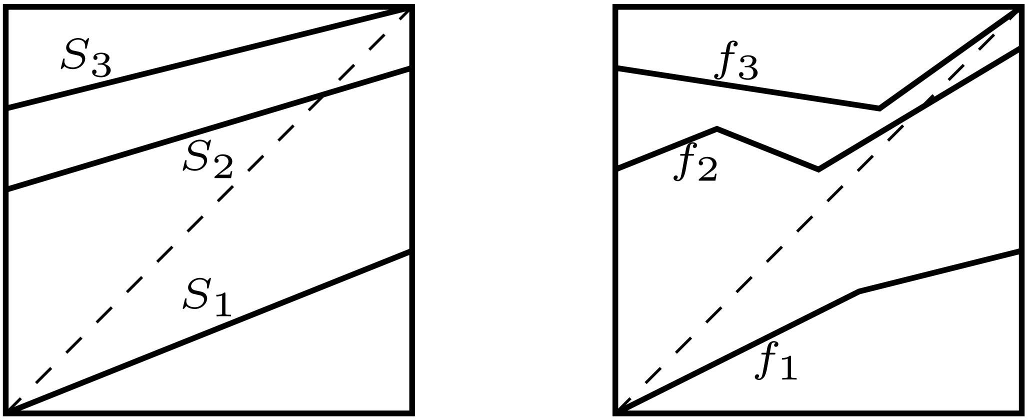
Definition 1.2

The attractor $\Lambda^{\mathcal{F}}$ of the IFS $\mathcal{F}$ is the unique non-empty compact set that satisfies

$$\Lambda^{\mathcal{F}} = \bigcup_{k=1}^{m} f_k(\Lambda^{\mathcal{F}}). \tag{1}$$

When all the f_i functions in $\mathcal{F}$ are linear, then $\mathcal{F}$ is called self-similar iterated function system. A lot is known about these systems, but overlapping constructions can be hard to handle.

The functions in our systems are only piecewise linear. Each function f_i might change slope at some point, thus they are not necessarily injective. Hence, overlapping structures are even more typical in our case.



Self-similar IFS
$\mathcal{S} = \{S_1, S_2, S_3\}$

CPLIFS
$\mathcal{F} = \{f_1, f_2, f_3\}$

Introducing piecewise linear iterated function systems

The natural pressure function

Our main results

One of the central questions in the field of fractal geometry is the dimension of the attractor of iterated function systems.

Definition 2.1

The Hausdorff dimension of a set $A \subset \mathbb{R}$ is defined as

$$\dim_{\mathrm{H}} A = \inf \left\{ s \geqslant 0 \middle| \, \forall \varepsilon > 0, \exists \{U_i\}_{i=1}^{\infty} : A \subset \bigcup_{i=1}^{\infty} U_i \;\&\; \sum_{i=1}^{\infty} |U_i|^s \leqslant \varepsilon \right\}.$$

The definition implies that the dimension of the attractor is connected to the yellow covering sum.

Without loss of generality, assume that the attractor $\Lambda^{\mathcal{F}}$ is contained in $[0,1]$, then by iteration

(2) $$\Lambda^{\mathcal{F}} = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in[m]^n} f_{i_1\dots i_n}([0,1]).$$

The sets $f_{i_1\dots i_n}([0,1])$ are called cylinder intervals, and they are the most natural cover of the attractor. We use these sets to define the natural pressure function of $\mathcal{F}$

(3) $$\Phi^{\mathcal{F}}(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum\nolimits_{i_1\dots i_n} |f_{i_1\dots i_n}([0,1])|^s$$

According to Barreira [1] the function $\Phi^{\mathcal{F}}(s)$ is strictly decreasing, continuous, positive at 0 and tends to $-\infty$ as $s \to \infty$. Thus we can define the natural dimension of $\mathcal{F}$ as

$$s_{\mathcal{F}} := (\Phi^{\mathcal{F}})^{-1}(0). \tag{4}$$

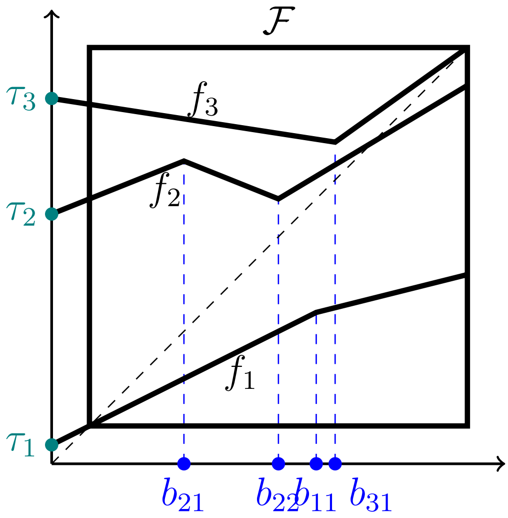
We showed that for the attractor of a typical CPLIFS the natural dimension equals to the Hausdorff dimension.

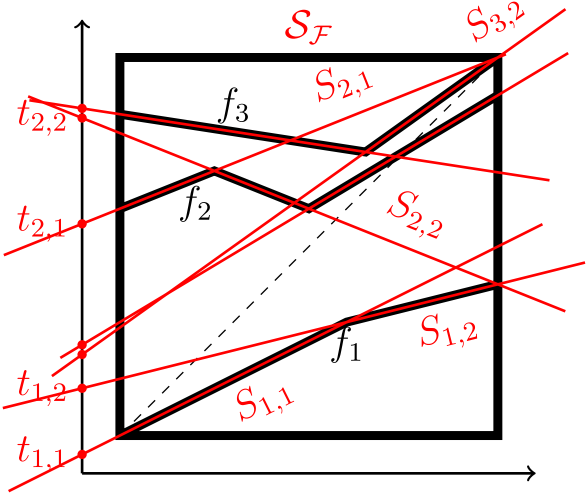
Introducing piecewise linear iterated function systems

The natural pressure function

Our main results

The generated self-similar IFS





Definition 3.1

We say that a small CPLIFS $\mathcal{F}$ is regular if its attractor $\Lambda_{\mathcal{F}}$ does not contain any of the breaking points.

For a regular CPLIFS $\mathcal{F}$ there is a smallest N such that

(5) there are no breaking points in $\bigcup_{i_1 \dots i_N} f_{i_1 \dots i_N}([0,1])$.

Theorem 3.2 (P., Simon [2])

For a $\dim_{\mathrm{P}}$*-typical small CPLIFS* $\mathcal{F}$ *we have*

(6)
$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = s_{\mathcal{F}}.$$

Theorem 3.3 (P.,Simon [2])

Let $\mathcal{F}$ *be a regular CPLIFS for which the generated self-similar IFS satisfies the Exponential Separation Condition (ESC). Then*

(7)
$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = s_{\mathcal{F}}.$$

Thank you for your attention!

References

[1] Luis M Barreira.
A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems.
Ergodic Theory and Dynamical Systems, 16(5):871–928, 1996.

[2] R Dániel Prokaj and Károly Simon.
Piecewise linear iterated function systems on the line of overlapping construction.
Nonlinearity, 2021.