PhD Thesis

On the dimension theory of piecewise linear iterated function systems

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Contents

Chapter 1

Introduction

Iterated Function Systems (IFS) on the line consist of finitely many strictly contracting self-mappings of $\mathbb{R}$. It was proved by Hutchinson [13] that for every IFS $\mathcal{F} = \{f_k\}_{k=1}^m$ there is a unique non-empty compact set $\Lambda^{\mathcal{F}}$ which is called the attractor of the IFS $\mathcal{F}$ and defined by

$$\Lambda^{\mathcal{F}} = \bigcup_{k=1}^{m} f_k(\Lambda^{\mathcal{F}}). \tag{1.1}$$

It is easy to see that for every IFS $\mathcal{F}$ there exists a unique "smallest" non-empty compact interval $I^{\mathcal{F}}$ which is sent into itself by all the mappings of $\mathcal{F}$:

$$I^{\mathcal{F}} := \bigcap \{J : J \subset \mathbb{R} \text{ compact interval with } f_k(J) \subset J,\ \forall k \in [m]\}, \tag{1.2}$$

where $[m] := \{1, \dots, m\}$. To guarantee that I is a non-degenerate interval, when the attractor of $\mathcal{F}$ is a single point we set

$$I := \left[\phi - \frac{1}{2}, \phi + \frac{1}{2}\right],$$

where ϕ is the common fixed point of the functions $f_1, \dots, f_m$. It is easy to see that

$$\Lambda^{\mathcal{F}} = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in[m]^n} I^{\mathcal{F}}_{i_1\dots i_n}, \tag{1.3}$$

where $I^{\mathcal{F}}_{i_1\dots i_n} := f_{i_1\dots i_n}(I^{\mathcal{F}})$ are the **cylinder intervals**, and we use the common shorthand notation $f_{i_1\dots i_n} := f_{i_1} \circ \dots \circ f_{i_n}$ for an $(i_1, \dots, i_n) \in [m]^n$. The following three one-dimensional IFS families appear on Figure 1.1.

(a) Self-similar IFS: $\mathcal{F} = \{f_i(x) = \rho_i x + t_i\}_{k=1}^m$, where $\rho_i \in (-1, 1) \setminus \{0\}$ and $t_i \in \mathbb{R}$.

(b) Hyperbolic IFS: $\mathcal{F} = \{f_1, \ldots, f_m\}$, where each $f_k : J \to J$ is a $C^{1+\varepsilon}(J)$ contracting self-mapping of a non-empty open interval $J \subset \mathbb{R}$. These iterated function systems are also often called self-conformal IFSs on the line.

(c) **Continuous Piecewise Linear Iterated Function Systems**:
These are IFSs of the form $\mathcal{F} = \{f_1, \ldots, f_m\}$, where $f_k : \mathbb{R} \to \mathbb{R}$ are continuous, piecewise linear (more precisely affine) contractions, not necessarily injective, with nonzero slopes that can be dissected to finitely many affine functions. We will often use the abbreviation CPLIFS.

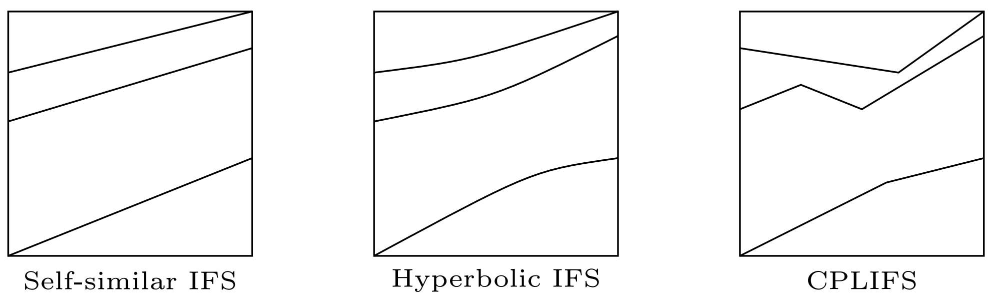

Figure 1.1

We note that the interval $I^{\mathcal{F}}$ introduced in (1.2) is the convex hull of the attractor in the first two cases, but not necessarily for non-injective CPLIFSs.

The t-dimensional Hausdorff measure of a Borel set $A \subset \mathbb{R}$ is

$$\mathcal{H}^t(A) := \sup_{\delta>0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : A \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| < \delta \right\} \right\}. \tag{1.4}$$

Let $\mathcal{F}$ be an arbitrary IFS on the line. We can give a natural upper bound on $\mathcal{H}^t(\Lambda^{\mathcal{F}})$ if we substitute the covering system made of the cylinder intervals $\left\{ I^{\mathcal{F}}_{i_1\ldots i_n} \right\}_{(i_1\ldots i_n)\in[m]^n}$ in the place of the most efficient cover in (1.4), just like in (1.3). For this particular system the right hand side of (1.4) is related to the sequence of sums $\{S_n^t\}_{n=1}^{\infty}$

$$S_n^t := \sum_{(i_1\ldots i_n)\in[m]^n} |I^{\mathcal{F}}_{i_1\ldots i_n}|^t.$$

Namely, if the exponential growth rate of $\{S_n^t\}_{n=1}^{\infty}$ is negative (positive), then it suggests that $\mathcal{H}^t(\Lambda^{\mathcal{F}})$ is equal to zero (infinity). Therefore the minimum of the dimension of the ambient space and the value of the exponent t for which this exponential growth rate is zero is a good natural guess for the Hausdorff dimension of $\Lambda^{\mathcal{F}}$. This motivates the introduction of the exponential growth rate

$$\Phi^{\mathcal{F}}(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I^{\mathcal{F}}_{i_1\dots i_n}|^s. \tag{1.5}$$

It is easy to see that we can obtain $\Phi^{\mathcal{F}}(s)$ above as a special case of the non-additive upper capacity topological pressure introduced by Barreira in [4, p. 5]. According to [4, Theorem 1.9], the zero of $\Phi^{\mathcal{F}}(s)$ is well defined

$$s_{\mathcal{F}} := (\Phi^{\mathcal{F}})^{-1}(0).$$

Barreira considered generalized Moran constructions [4, Section 2.1.2]. Condition (b) of such a construction requires that the cylinder intervals $\left\{I^{\mathcal{F}}_{i_1\dots i_n}\right\}$ are sufficiently well separated. This separation assumption does not hold for the constructions considered in this paper, thus not all of his results apply. In the last inequality of [4, Theorem 2.1 part (a)] Barreira proves that the upper box dimension of $\Lambda^{\mathcal{F}}$ is less than or equal to the root of the non-additive upper capacity topological pressure. It is easy to see that its proof does not require any separation conditions on the cylinder intervals. That is

Corollary 1.0.1 (Barreira). *For any IFS $\mathcal{F}$ on the line*

$$\overline{\dim}_{\mathrm{B}}\Lambda^{\mathcal{F}} \le s_{\mathcal{F}.}$$

For a given IFS $\mathcal{F}$ on the line we name $s_{\mathcal{F}}$ the **natural dimension of the system**. It is easy to see that in the self-simiar case $s_{\mathcal{F}}$ is the solution of the equation

$$\sum_{k=1}^{m} |\rho_k|^{s_{\mathcal{F}}} = 1.$$

We call $s_{\mathcal{F}}$ the similarity dimension in this case. If $\mathcal{F}$ is a hyperbolic system, then $s_{\mathcal{F}}$ is the root of the so-called pressure formula (see [21]). In both cases the Open Set Condition (OSC) implies that

$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \min\{s_{\mathcal{F}}, 1\}. \tag{1.6}$$

The cylinder intervals of the attractor of a system that satisfies the OSC are well separated (see [8, p. 35]). However, less strict separation conditions might lead to the same result. In the self-similar case the celebrated Hochman Theorem [9, Theorem 1.1] yields that (1.6) also follows from the Exponential Separation Condition (ESC).

1.1 Continuous Piecewise Linear IFSs

The IFSs we consider in this thesis are consisting of piecewise linear (more precisely affine) functions. Thus their derivatives might change at some points, but they are linear over given intervals of $\mathbb{R}$. We always assume that the functions are continuous, piecewise linear, strongly contracting with non-zero slopes, and that the slopes can only change at finitely many points. This setup enables us to investigate the case of non-injective functions as well, which is not achievable with either hyperbolic or self-similar systems.

For a special case of CPLIFSs, one may obtain results on the dimension of the attractor using the theory of F. Hofbauer [12] and P. Raith [24]. In particular, a family of CPLIFSs satisfying the one dimensional version of the rectangular open set condition.

Definition 1.1.1. *We say that a CPLIFS $\mathcal{F} = \{f_k\}_{k=1}^m$ satisfies the* ***Interval Open Set Condition*** *(IOSC) if the first cylinder intervals $\{I_k^{\mathcal{F}}\}_{k=1}^m$ are pairwise disjoint.*

Take a CPLIFS $\mathcal{F} = \{f_k\}_{k=1}^m$ of injective functions that satisfies the IOSC. Each f_k function can be considered as the local inverse over $I_k^{\mathcal{F}}$ of a strictly expanding map on $\mathbb{R}$. Therefore, the Hausdorff dimension of the attractor of such systems is equal to the root of a topological pressure function [24]. Further, according to [12], for this special CPLIFS family the Hausdorff and the box dimensions of the attractor are also equal. We will introduce the topological pressure function that Raith and Hofbauer used, and show how it relates to the natural pressure function in Section 2.2.

In general, when we have no information about the possible overlapping of the cylinders, we can only claim that (1.6) holds in some sense typically. Instead of a particular IFS, it is natural to consider its so-called translation family.

Definition 1.1.2. *For every $\boldsymbol{\tau} = (\tau_1, \dots, \tau_m) \in \mathbb{R}^m$ we define the translation family of $\mathcal{F}$ by*

$$\{\mathcal{F}^{\boldsymbol{\tau}}\}_{\boldsymbol{\tau}\in\mathbb{R}^m}, \text{ where } \mathcal{F}^{\tau} = \{f_1^{\boldsymbol{\tau}}, \dots, f_m^{\boldsymbol{\tau}}\}$$

and $f_k^{\boldsymbol{\tau}}(x) := f_k(x) + \tau_k$ for a $k \in [m]$. Moreover, for an $\mathbf{i} = (i_1, \dots, i_n) \in \Sigma^*$ we define the function

$$f_{\mathbf{i}}^{\boldsymbol{\tau}}(x) := f_{i_1}^{\boldsymbol{\tau}} \circ \cdots \circ f_{i_n}^{\boldsymbol{\tau}}(x).$$

The classical results [6], [7] and [26] about translation families of some IFSs read like this: For typical (in some sense) translations $\boldsymbol{\tau}$, the Hausdorff dimension of the translated attractor $\Lambda^{\boldsymbol{\tau}}$ is equal to its natural dimension.

Here the sense of typicality depends on the family of the IFS considered. For example, for hyperbolic IFS on the line the typicality above means "typical with respect to the m-dimensional Lebesgue measure" [26] (at least in the case when all contractions are less than $\frac{1}{2}$). In this case we say that the Hausdorff dimension and the natural dimension coincide for Lebesgue typical translations. On the other hand, Hochman [10] proved a much stronger theorem for the translation family of self-similar IFSs that we call **Multi Parameter Hochman Theorem** (see Theorem 2.3.6). This theorem implies that the exceptional set of translations $\boldsymbol{\tau}$ for which the ESC does not hold for $\mathcal{F}^{\boldsymbol{\tau}}$ (and consequently $\dim_{\mathrm{H}} \Lambda^{\boldsymbol{\tau}}$ is different from the similarity dimension), has packing dimension (and in this way also Hausdorff dimension) less than or equal to $m-1$. Remember that the dimension of the parameter space was equal to m. Motivated by this, we introduce the following terminology related to more general IFS families on the line:

Terminology: Let $\left\{\mathcal{F}^{\boldsymbol{\lambda}}\right\}_{\boldsymbol{\lambda}\in U}$ be a family of IFSs on the line where the parameter set U is an open subset of $\mathbb{R}^d$ for some $d \geq 1$. We say that a **property** $\mathfrak{P}$, which makes sense for all elements of this family, **holds** $\dim_{\mathrm{P}}$-**typically** if the exceptional set E of those parameters $\boldsymbol{\lambda} \in U$ for which $\mathfrak{P}$ does not hold satisfies $\dim_{\mathrm{P}} E < d$. That is the packing dimension of the exceptional set $E \subset U$ is strictly smaller than the dimension of the parameter space U.

1.2 Structure of the Thesis

The aim of this dissertation is to summarize the theory of continuous piecewise linear iterated function systems on the line. We outline here the structure of the thesis and all the new results presented in the upcoming sections.

After the introduction, we rigorously describe the parameter space of continuous piecewise linear iterated function systems and the notion of packing dimension-typicality. In the same chapter, we also show the connection between our natural pressure function and the topological pressure function of the thermodynamical formalism of smooth dynamical systems which is often used to calculate the Hausdorff dimension of the underlying attractor.

In the proofs of our results, we apply theorems from the field of self-simlar and self-similar graph-directed iterated function systems. These theorems are listed in Section 2.3.

After the preliminaries, we first turn our attention to the special case of small CPLIFSs. We call a CPLIFS $\mathcal{F} = \{f_k\}_{k\in[m]}$ **small** if the slopes of its functions are all bounded from above by certain small constants (see Definition 3.0.1). If the piecewise linear functions f_k do not change their slopes on the attractor of $\mathcal{F}$, we say that the CPLIFS is **regular** (see Definition 3.0.2). We will show that the attractor of a regular IFS can be described using an appropriately defined self-similar graph-directed IFS.

In Chapter 3, we give our main result on small CPLIFSs (Theorem 3.2.1). Our aim is to prove that $\dim_{\mathrm{P}}$-*typically*

$$\mathcal{F} \text{ is small} \Longrightarrow \mathcal{F} \text{ is regular}, \quad \& \quad \mathcal{F} \text{ is regular} \Longrightarrow \dim_{\mathrm{H}} \Lambda = s_{\mathcal{F}}.$$

These implications are addressed in Theorem 3.2.2 and Proposition 3.2.3, respectively. To verify them, we will prove some simple properties of regular CPLIFSs (see Lemma 3.2.9 and Lemma 3.2.6).

In Chapter 4 the general case is discussed. Here we prove that the fractal dimensions of the attractor of a CPLIFS are typically equal to the minimum of 1 and the natural dimension (Theorem 4.0.1).

To investigate the dynamical structure of our attractors, we introduce Markov diagrams for continuous piecewise linear iterated function systems. As Lemma 4.1.8 suggests, these diagrams are directly connected to the natural pressure function. That is, we can approximate the natural dimension of our IFS by constructing a series of subsystems (Lemma 4.2.1). We close this chapter with a case analysis proving that this approximation method applies for a packing dimension typical CPLIFS (Proposition 4.3.1).

Let μ be a Borel probability measure on $\mathbb{R}$. The Hausdorff dimension of μ is defined as

$$\dim_{\mathrm{H}} \mu := \inf \{\dim_{\mathrm{H}} E : \mu(E) > 0\}.$$

By definition, the Hausdorff dimension of the underlying attractor $\dim_{\mathrm{H}} \Lambda$ is always bigger or equal to the Hausdorff dimension of a Borel measure supported on Λ. If our CPLIFS is regular, we can always find an invariant ergodic measure of full Hausdorff dimension. We detail how to obtain this measure in Section 3.1.5. While the construction we give does not apply to a general CPLIFS, we show in Section 4.2 that the supremum of the Hausdorff dimension of invariant ergodic measures supported on the attractor is typically $\dim_{\mathrm{H}} \Lambda$.

$$\dim_{\mathrm{H}} \Lambda = \sup\{\dim_{\mathrm{H}} \mu : \mu \text{ is ergodic and invariant}, \operatorname{supp}(\mu) \subset \Lambda\}. \quad (1.7)$$

In general, (1.7) may not hold. Das and Simmons [5] proved that there are iterated function systems, where all invariant measures have Hausdorff dimension bounded away from the Hausdorff dimension of the attractor.

One can always represent a regular CPLIFS as a self-similar graph-directed iterated function system. Chapter 5 discusses the special case of non-regular CPLIFSs that has a self-similar graph-directed representation. We construct the directed graphs and give a recursive formula for the natural dimension of the attractor assuming that the piecewise linear functions can only change slopes at their fixed points.

The last chapter is dedicated to continuity results. We show that the natural dimension is always continuous with respect to the defining parameters of a CPLIFS under a really weak separation condition (Theorem 6.0.3). As an application of this result, we prove that the attractor of a CPLIFS with positive slopes typically has positive Lebesgue measure if the natural dimension is bigger than 1 (Theorem 6.2.1).

Chapter 2

Preliminaries

2.1 Parametrizing CPLIFSs

We fix a number $m \geq 2$, and use it as the number of functions in a CPLIFS throughout the thesis. Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS. We write $l(k)$ for the number of breaking points of f_k for $k \in [m]$, and we say that the **type of the CPLIFS** is the vector

$$\boldsymbol{\ell} = (l(1), \dots, l(m)).$$

For example the type of the CPLIFS on Figure 2.1 is $\boldsymbol{\ell} = (1, 2)$. If $\mathcal{F}$ is a CPLIFS of type $\boldsymbol{\ell}$, then we write

$$\mathcal{F} \in \mathrm{CPLIFS}_{\boldsymbol{\ell}}.$$

We write $B(k)$ for the set of breaking points of f_k, and we denote its elements by $b_{k,1} < \cdots < b_{k,l(k)}$. Let $L := \sum_{k=1}^{m} l(k)$ be the **total number of breaking points** of the functions of $\mathcal{F}$ counted with multiplicity if some of the breaking points of two different elements of $\mathcal{F}$ coincide. We arrange all the breaking points in an L dimensional vector $\mathfrak{b} \in \mathbb{R}^L$ in a way described below. First we partition $[L] = \{1, \dots, L\}$ into blocks of length $l(k)$ for $k \in [m]$. The k-th block is

$$L^k := \left\{ p \in \mathbb{N} : 1 + \sum_{j=1}^{k-1} l(j) \leq p \leq \sum_{j=1}^{k} l(j) \right\}$$

where $\sum_{j=1}^{k-1}$ is meant to be 0 when $k = 1$. We use this convention without further mentioning it throughout the thesis. The breaking points of f_k occupy the components belonging to the block L^k in increasing order. That is

$$\mathfrak{b} = (\underbrace{b_{1,1}, \dots, b_{1,l(1)}}_{L^1}, \underbrace{b_{2,1}, \dots, b_{2,l(2)}}_{L^2}, \dots, \underbrace{b_{m,1}, \dots, b_{m,l(m)}}_{L^m}). \tag{2.1}$$

The set of breaking point vectors $\mathfrak{b}$ for a type $\boldsymbol{\ell}$ CPLIFS is

$$\mathfrak{B}^{\boldsymbol{\ell}} := \left\{ \mathbf{x} \in \mathbb{R}^L : x_i < x_j \text{ if } i < j \text{ and } \exists k \in [m] \text{ with } i, j \in L^k \right\}.$$

The $l(k)$ breaking points of the piecewise linear continuous function f_k determines the $l(k)+1$ **intervals of linearity** $J^{\mathfrak{b}}_{k,i}$, among which the first and the last are actually half lines:

$$J_{k,i} := J^{\mathfrak{b}}_{k,i} := \begin{cases} (-\infty, b_{k,1}), & \text{if } i = 1; \\ (b_{k,i-1}, b_{k,i}), & \text{if } 2 \le i \le l(k); \\ (b_{k,l(k)}, \infty), & \text{if } i = l(k)+1. \end{cases}$$

The derivative of f_k exists on $J_{k,i}$ and is equal to the constant

$$\rho_{k,i} :\equiv f'_k|_{J_{k,i}}. \tag{2.2}$$

We arrange the contraction ratios $\rho_{k,i} \in (-1,1) \setminus \{0\}$ into a vector $\boldsymbol{\rho}$ in an analogous way as we arranged the breaking points into a vector in (2.1), but taking into account that there is one more contraction ratio for each f_k than breaking point:

$$\boldsymbol{\rho} := \boldsymbol{\rho}_{\mathcal{F}} := (\underbrace{\rho_{1,1}, \dots, \rho_{1,l(1)+1}}_{\widetilde{L}^1}, \dots, \underbrace{\rho_{m,1}, \dots, \rho_{m,l(m)+1}}_{\widetilde{L}^m}) \in ((-1,1) \setminus \{0\})^{L+m},$$

where

$$\widetilde{L}^k := \left\{ p \in \mathbb{N} : 1 + \sum_{j=1}^{k-1} (1 + l(j)) \le p \le \sum_{j=1}^{k} (1 + l(j)) \right\}.$$

We call $\boldsymbol{\rho}$ the **vector of contractions**. The set of all possible values of $\boldsymbol{\rho}$ for an $\mathcal{F} \in \text{CPLIFS}_{\boldsymbol{\ell}}$ is

$$\mathfrak{R}^{\boldsymbol{\ell}} := \left\{ \boldsymbol{\rho} \in ((-1,1) \setminus \{0\})^{L+m} : \forall k \in [m],\, \forall i, i+1 \in \widetilde{L}^k, \rho_i \neq \rho_{i+1} \right\},$$

where $\boldsymbol{\rho} = (\rho_1, \dots, \rho_{L+m})$. We introduce the following notation for the biggest and smallest slope in the IFS.

$$\rho_k := \max_{i \in [l(k)+1]} |\rho_{k,i}|, \quad \rho_{\max} := \max_{k \in [m]} \rho_k, \text{ and } \rho_{\min} := \min_{k \in [m]} \min_{i \in [l(k)+1]} |\rho_{k,i}| \tag{2.3}$$

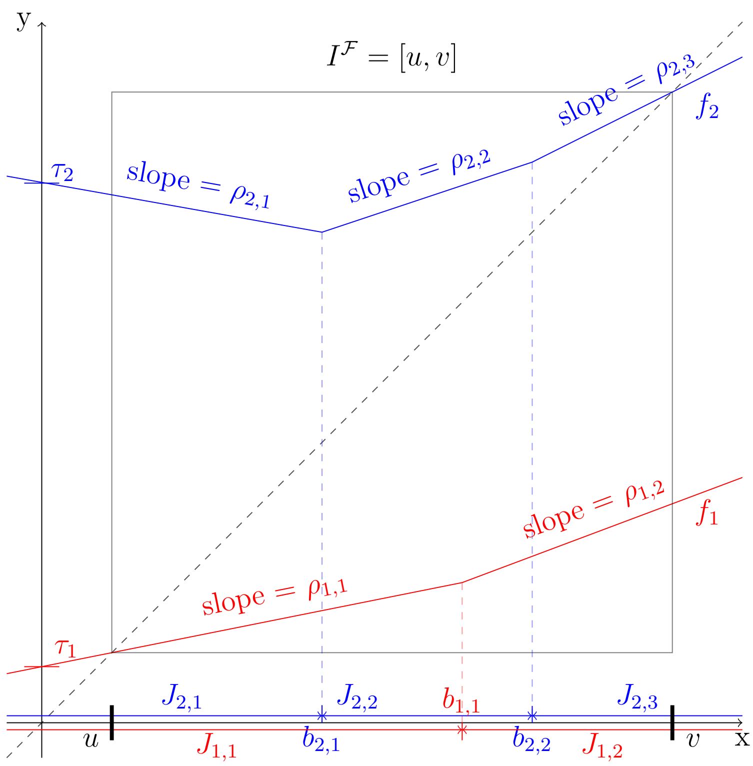

Figure 2.1: A general CPLIFS with the related notations.

Moreover, let $\rho_{k_1\dots k_n} := \rho_{k_1}\cdots\rho_{k_n}$. Clearly,

$$|f'_{k_1\dots k_n}(x)| \leq \rho_{k_1\dots k_n}, \text{ for all } x.$$

Finally, we write

$$\tau_k := f_k(0), \text{ and } \boldsymbol{\tau} := (\tau_1, \dots, \tau_m) \in \mathbb{R}^m.$$

So, the parameters that uniquely determine an $\mathcal{F} \in \mathrm{CPLIFS}_{\boldsymbol{\ell}}$ can be organized into a vector

$$\boldsymbol{\lambda} = (\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho}) \in \boldsymbol{\Gamma}^{\boldsymbol{\ell}} := \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \times \mathfrak{R}^{\boldsymbol{\ell}} \subset \mathbb{R}^L \times \mathbb{R}^m \times \mathbb{R}^{L+m} = \mathbb{R}^{2L+2m}.$$

It is easy to see that $\boldsymbol{\lambda}$ indeed determines a CPLIFS uniquely. For instance, assuming that all the breaking points in the system are positive, we have the following formula for the functions of $\mathcal{F}$:

$$f_k(x) = \tau_k + \sum_{j=1}^{l(k)} (b_{k,j} - b_{k,j-1})\rho_{k,j}\mathbb{1}\{b_{k,j} \leq x\} + \sum_{j=1}^{l(k)+1} \rho_{k,j}(x - b_{k,j-1})\mathbb{1}\{x \in J_{k,j}\},$$

for any $k \in [m]$ and $x \geq 0$, where we used the notation $\forall k \in [m] : b_{k,0} = 0$ to make the formula more compact.

For a $\boldsymbol{\lambda} \in \boldsymbol{\Gamma}^{\boldsymbol{\ell}}$ we write $\mathcal{F}^{\boldsymbol{\lambda}}$ for the corresponding CPLIFS, $\Lambda^{\boldsymbol{\lambda}}$ for its attractor, and $s_{\boldsymbol{\lambda}}$ for its natural dimension. Similarly, for an $\mathcal{F} \in \text{CPLIFS}_{\boldsymbol{\ell}}$ we write $\boldsymbol{\lambda}(\mathcal{F})$ for the corresponding element of $\boldsymbol{\Gamma}^{\boldsymbol{\ell}}$. With the help of these notations we can taylor the terminology of $\dim_{\mathrm{P}}$-typicality for CPLIFSs.

Terminology 2.1.1. *Let $\mathfrak{P}$ be a property that makes sense for every continuous piecewise linear iterated function system. For a contraction vector $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}$ we consider the (exceptional) set*

$$E^{\boldsymbol{\rho}}_{\mathfrak{P},\boldsymbol{\ell}} =: \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m : \mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \text{ does not have property } \mathfrak{P}\right\}. \tag{2.4}$$

*We say that **property $\mathfrak{P}$ holds $\dim_{\mathrm{P}}$-typically** if for all type $\boldsymbol{\ell}$ and for all $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}$ we have*

$$\dim_{\mathrm{P}} E^{\boldsymbol{\rho}}_{\mathfrak{P},\boldsymbol{\ell}} < L + m,$$

where $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and $L = \sum\limits_{k=1}^{m} l(k)$ as above.

2.2 Relation to the topological pressure

Let (X, d) be a compact metric space, $G : X \to X$ be a continuous mapping, and $g \in \mathcal{C}(X, \mathbb{R})$ be a continuous real valued function. The classical topological pressure is defined as

$$p(R, G, g) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \sum_{x \in E} \exp\left(\sum_{j=0}^{n-1} g(G^j x)\right), \tag{2.5}$$

where the supremum is taken over all (n, ε)-separated subsets E of R. A set $E \subset R$ is (n, ε)-separated, if for every $x \neq y \in E$ there exists a $j \in \{0, 1, \dots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$, where d is the metric on X which induces the order topology.

The topological pressure function is often used to calculate the Hausdorff dimension of the attractor of iterated function systems of C^2 functions [8, Theorem 5.3]. To motivate our result on the equality of dimensions, we are going to show that the natural dimension coincides with the root of the topological pressure function, using the case of injective CPLIFSs satisfying the IOSC.

Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be an injective CPLIFS that satisfies the IOSC, and without loss of generality assume $I^{\mathcal{F}} = [0, 1]$. We write $I_{i_1 \dots i_k} = f_{i_1 \dots i_k}(I)$

and Λ for the attractor of $\mathcal{F}$. Recall, that we denoted with $B(k)$ the set of breaking points of f_k. We simply write $f_k(B(k)) := \{f_k(x) : x \in B(k)\}$ for the set of the images of the breaking points of f_k, $k \in [m]$.

Let $T : \cup_{k\in[m]} I_k \to [0,1]$ be defined as follows

$$\forall k \in [m] \ : \ (T|_{I_k})^{-1} = f_k.$$

An injective CPLIFS that satisfies the IOSC and its associated expansive mapping T are depicted on Figure 2.2.

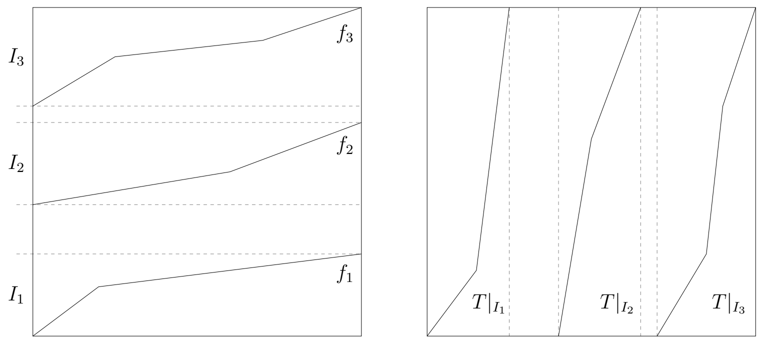

Figure 2.2

Write W_0 for the set of points where the derivative of T is not defined. Hence $W_0 = \bigcup_{k=1}^{m} (\{f_k(0), f_k(1)\} \cup f_k(B(k)))$. We define W as the set of preimages of the elements of W_0 except 0 and 1:

$$W := \left(\bigcup_{i=0}^{\infty} T^{-i}(W_0 \setminus \{0,1\}) \right) \setminus \{0,1\}.$$

Now let

$$R = \bigcap_{n=0}^{\infty} \left([0,1] \setminus T^{-n} \left([0,1] \setminus \bigcup_{k=1}^{m} I_k \right) \right).$$

Thus R contains all the points whose orbit will never leave the union of the first cylinders as we iterate T. Observe that $R = \Lambda$.

Instead of $[0,1]$ we will work on a different metric space, obtained by doubling some points, that we denote with $[0,1]_W$. Namely, following [24, p. 41], we double all elements of W, and equip this new space with the metric that induces the order topology. We call this new complete metric space **Doubled points topology**. We write R_W for the closure of $R \setminus W$ in the doubled points topology. Let T_W be the unique, continuous extension of our expanding map T in this new metric space. Similarly, for a piecewise constant function ψ let ψ_W denote the unique continuous function for which $\psi_W(x) = \psi(x)$, $\forall x \in [0,1] \setminus (W \cup \{0,1\})$. We call ψ_W the completion of ψ.

In the doubled points topology $\psi_s(x) = -s \log |T'(x)|$ has a continuous completion that we denote by $\psi_{s,W}$. Thus the pressure function (2.5) is well defined for $X = R_W$, $G = T_W$, $g = \psi_{s,W}$, where the choice of $s \in \mathbb{R}$ is arbitrary. This is the reason why we work on this new topological space.

According to [24, Lemma 3] the map

$$P_{top}(s) := p(R_W, T_W, \psi_{s,W}) \tag{2.6}$$

is continuous and strictly decreasing. Moreover, the root of this map s_{top} coincides with the Hausdorff dimension of R_W. Since R and R_W only differs in countably many points $\dim_H R = \dim_H R_W$. We call the map $P_{top}(s)$ **Topological Pressure Function**. As a consequence we obtain

Lemma 2.2.1. *Let $\mathcal{F}$ be a CPLIFS on the line that satisfies the IOSC, and denote its attractor with Λ. Then*

$$\dim_{\mathrm{H}} \Lambda = s_{\text{top}},$$

where s_{top} is the unique root of the topological pressure defined in (2.6).

For a $\mathcal{F} = \{f_i\}_{i=1}^m$ CPLIFS let $(s_n)_{n\geq 1}$ be the unique sequence for which

$$\sum_{i_1 \dots i_n} |I_{i_1 \dots i_n}|^{s_n} = 1 \tag{2.7}$$

holds for every $n \geq 1$. The following lemma shows that $s = \limsup_{n\to\infty} s_n$ equals to the root of the natural pressure $s_{\mathcal{F}}$. Recall that we write $\Lambda^{\mathcal{F}}$ for the attractor of the CPLIFS $\mathcal{F}$.

Lemma 2.2.2. *For a $\mathcal{F}$ CPLIFS defined on* $[0,1]$, *let $(s_n)_{n\geq 1}$ be the unique series that satisfies 2.7. Then the following holds*

$$s_{\mathcal{F}} = \limsup_{n\to\infty} s_n,$$

where $s_{\mathcal{F}}$ is the root of the natural pressure function $\Phi^{\mathcal{F}}$.

Proof. Recall that we denote the smallest and largest contraction ratio of $\mathcal{F}$ by $\rho_{\min}$ and $\rho_{\max}$ respectively, and fix an arbitrary $n \geq 1$.

For a given length n word $i_1 \dots i_n$ and an arbitrary s we can write $|I_{i_1\dots i_n}|^s = |I_{i_1\dots i_n}|^{s_n} \cdot |I_{i_1\dots i_n}|^{s-s_n}$ to obtain the estimates

$$\rho_{\min}^{n(s-s_n)} \cdot |I_{i_1\dots i_n}|^{s_n} \leq |I_{i_1\dots i_n}|^s \leq \rho_{\max}^{n(s-s_n)} \cdot |I_{i_1\dots i_n}|^{s_n} \tag{2.8}$$

Since by definition $\sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^{s_n} = 1$, equation (2.8) implies

$$(s - s_n)\log \rho_{\min} \leq \frac{1}{n}\log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s \leq (s - s_n)\log \rho_{\max}$$

Reordering the inequalities we obtain

$$\frac{\frac{1}{n}\log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s}{\log \rho_{\max}} \leq (s - s_n) \leq \frac{\frac{1}{n}\log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s}{\log \rho_{\min}}$$

If we choose s to be equal to $s_{\mathcal{F}}$, then taking the limit superior of each side yields $\limsup_{n\to\infty} s_{\mathcal{F}} - s_n = 0$.

□

Using this lemma we show that for an injective CPLIFS $\mathcal{F}$ the root of the natural pressure (1.5) coincides with the root of the topological pressure function (2.6) of the associated expanding map.

Lemma 2.2.3. *Let $\mathcal{F}$ be an injective CPLIFS that satisfies the IOSC. Write s_{top} for the root of the topological pressure function* (2.6)*, and $s_{\mathcal{F}}$ for the root of the natural pressure function* (1.5)*. Then*

$$s_{top} = s_{\mathcal{F}}.$$

Proof. Without applying Corollary 1.0.1, we first prove that the Hausdorff dimension of the attractor is smaller than or equal to the natural dimension, using the sequence $(s_n)_{n\geq 1}$ defined in (2.7). Fix an $\varepsilon > 0$.

By Lemma 2.2.2 $s_{\mathcal{F}} = \limsup_{n\to\infty} s_n$, hence $\exists N$ such that $\forall n > N : s_n < s_{\mathcal{F}} + \frac{\varepsilon}{2}$. Thus by (2.7) we have

$$\sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^{s_{\mathcal{F}}+\varepsilon} < \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^{s_n+\frac{\varepsilon}{2}} \leq \rho_{\max}^{\frac{n\varepsilon}{2}} \to 0, \text{ as } n \to \infty.$$

It implies that $\mathcal{H}^{s_{\mathcal{F}}+\varepsilon}(\Lambda^{\mathcal{F}}) = 0$ for each $\varepsilon > 0$, where $\mathcal{H}^s$ stands for the s-dimensional Hausdorff measure. By the definition of the Hausdorff dimension, we obtain

$$\dim_H \Lambda^{\mathcal{F}} \leq s_{\mathcal{F}}$$

According to Lemma 2.2.1 $\dim_H \Lambda^{\mathcal{F}} = s_{top}$, so we already proved

$$s_{top} \leq s_{\mathcal{F}}.$$

To prove the other direction, we first need to reformalise the pressure function $P_{top}(s)$, to see how it relates to the natural pressure $\Phi^{\mathcal{F}}(s)$.

Recall that we assumed that the IOSC holds, which means that all of the first level cylinders are separated by some positive distance $D > 0$. Fix an $D > \varepsilon > 0$, and let $N(\varepsilon)$ be sufficiently big such that

$$\max_{i_1,\ldots,i_{N(\varepsilon)}} |I_{i_1,\ldots,i_{N(\varepsilon)}}| < \varepsilon.$$

We will show that by choosing one element of each level $N(\varepsilon)$ cylinder we obtain an $(N(\varepsilon), \varepsilon)$-separated set.

For any two N length words $\mathbf{i} = \{i_1, \ldots i_N\}$, $\mathbf{j} = \{j_1, \ldots j_N\}$ let $|\mathbf{i} \wedge \mathbf{j}| = \min\{k - 1 : \ i_k \neq j_k\}$. Thus if we iterate $|\mathbf{i} \wedge \mathbf{j}|$ times T over the cylinders $I_{i_1,\ldots,i_N}$, $I_{j_1,\ldots,j_N}$, the images will fall into different first level cylinders. More formally

$$d(T^{|\mathbf{i}\wedge\mathbf{j}|} I_{\mathbf{i}},\ T^{|\mathbf{i}\wedge\mathbf{j}|} I_{\mathbf{j}}) > D > \varepsilon.$$

Therefore by choosing one element from each $N(\varepsilon)$ level cylinder, we obtain an $(N(\varepsilon), \varepsilon)$-separated subset that we denote by $I_{sep}^{N(\varepsilon)}$. We require $\forall x \in I_{sep}^{N(\varepsilon)}$ to maximize the derivative of T over the $N(\varepsilon)$ cylinder that contains x. We can make this constraint, since any choice of elements will do. Remember that we use the doubled points topology introduced in [24, p. 41], so T' is well defined at every $x \in [0, 1]_W$.

We can define I_{sep}^n similarly for any $n > N(\varepsilon)$. We substitute these sets into the topological pressure to gain a lower bound. We use the notation $\psi_s(x) := -s \log |T'(x)|$ to make the formulas more concise.

$$\begin{aligned} P_{top}(s) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left(\sum_{j=0}^{n-1} \psi_s(T^j x)\right) \\ &\geq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in I_{sep}^n} \exp\left(\sum_{j=0}^{n-1} \psi_s(T^j x)\right) \\ &\geq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}:\, \Pi(\mathbf{i})\in I_{sep}^n} \exp\left(\sum_{j=0}^{n-1} |I_{\mathbf{i}}|^s\right) = \Phi^{\mathcal{F}}(s), \end{aligned}$$

where in the last inequality we substituted $\psi_s(x) = -s \log |T'(x)|$. For $s = s_{\mathcal{F}}$, the right hand side is equal to 0. The pressure function $P_{top}(s)$ is strictly decreasing, thus its unique zero s_{top} must be bigger or equal to $s_{\mathcal{F}}$. We just obtained

$$s_{\mathcal{F}} \leq s_{top}.$$

□

We note that the statement of Lemma 2.2.3 also holds with a weakened version of the IOSC. Namely, it is enough to assume that the interiors of the first cylinder intervals $f_i(I), i \in [m]$ are disjoint. The proof is analogous, but in the construction of the (N, ε)-separated subset the choice of ε and N will depend on the slopes of the functions of $\mathcal{F}$.

2.3 Self-similar and self-similar graph-directed IFSs

Here we summarize the relevant results from the theory of self-similar IFSs. Let $\mathcal{F} = \{f_k\}_{k=1}^M$ be an IFS on $\mathbb{R}$ and Λ be the attractor of $\mathcal{F}$. As usual, the natural projection $\Pi : \Sigma \to \Lambda$ is defined by

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} f_{\mathbf{i}|n}(0).$$

We write σ for the left-shift on Σ. For the definition of invariance, ergodicity, and entropy (denoted by $h(\mu)$) of the measure μ we refer the reader to Walters' book [27]. If the measure μ is ergodic and the derivatives of all of the mappings f_i are continuous, we define the Lyapunov exponent of μ by

$$\chi(\mu) := \int \log |f'_{i_1}(\Pi(\sigma\mathbf{i}))| d\mu(\mathbf{i}). \tag{2.9}$$

2.3.1 Self-similar IFSs on the line

In the special case when all f_i are similarities (the slopes are constants) $\mathcal{F}$ is called self-similar IFS. In this case for all $i \in [M] := \{1, \dots, M\}$ the mappings f_i, can be presented in the form

$$f_i(x) = r_i x + t_i, \text{ where } r_i \in (-1, 1) \setminus \{0\} \text{ and } t_i \in \mathbb{R}.$$

The simplest guess for the dimension of the attractor Λ can be expressed in terms of the **similarity dimension** $\dim_{\mathrm{S}} \Lambda$, which is defined as the solution s of the equation $\sum\limits_{K=1}^{M} |r_k|^s = 1$. That is

$$\dim_{\mathrm{H}} \Lambda \leq \min \{1, \dim_{\mathrm{S}} \Lambda\}.$$

It follows from a theorem of Hochman's (Theorem 2.3.3) that in some sense typically we have equality above. For an $\bar{\imath} := (i_1, \dots, i_n)$ we define

$$r_{\bar{\imath}|_0} := 1, \quad r_{\bar{\imath}} = r_{i_1\dots i_n} = r_{i_1} \cdot \ldots \cdot r_{i_n} \text{ and } t_{\bar{\imath}} := \sum_{k=1}^{n} t_{i_k} r_{\bar{\imath}|_k}.$$

Clearly, we have $f_{\overline{\imath}}(x) = r_{\overline{\imath}}x + t_{\overline{\imath}}$ and

$$\Pi(\mathbf{i}) = \sum_{n=1}^{\infty} r_{i_1 \dots i_{n-1}} t_{i_n} = \lim_{n\to\infty} t_{\mathbf{i}|_n}.$$

If μ is a measure on Σ then we write $\Pi_*\mu$ for the push forward measure of μ. That is $\Pi_*\mu(E) = \mu(\Pi^{-1}E)$. Let μ be a σ-invariant ergodic measure on Σ. Then the Lyapunov exponent is

$$\chi(\mu) = \sum_{k=1}^{m} \mu[k] \log r_k.$$

For a probability vector $\mathbf{p} := (p_1, \dots, p_M)$ we define the measures:

$$\mu_{\mathbf{p}}([i_1, \dots, i_n]) := p_{i_1} \cdots p_{i_n}, \text{ and } \nu_{\mathbf{p}} := \Pi_* \mu_{\mathbf{p}}.$$

We say that $\nu_{\mathbf{p}}$ is a **self-similar measure**. The simplest guess for the Hausdorff dimension of a self-similar measure $\nu_{\mathbf{p}}$ is

$$\dim_{\mathrm{S}} \nu_{\mathbf{p}} := \frac{h(\nu_{\mathbf{p}})}{\chi(\nu_{\mathbf{p}})} = \frac{\sum\limits_{k=1}^{M} p_k \log p_k}{\sum\limits_{k=1}^{M} p_k \log r_k}.$$

$\dim_{\mathrm{H}} \nu_{\mathbf{p}} \leq \dim_{\mathrm{S}} \nu_{\mathbf{p}}$ and a theorem of Hochman states that typically we have equality. We say that $\dim_{\mathrm{S}} \nu_{\mathbf{p}}$ is the **similarity dimension of the measure** $\nu_{\mathbf{p}}$.

Hochman [9] introduced the notion of exponential separation for self-similar IFSs. To state it, first we need to define the distance of two similarity mappings $g_1(x) = \varrho_1 x + \tau_1$ and $g_2(x) = \varrho_2 x + \tau_2$, $\varrho_1, \varrho_2 \in (-1, 1) \setminus \{0\}$, on $\mathbb{R}$. Namely,

$$\mathrm{dist}\,(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } \varrho_1 = \varrho_2; \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 2.3.1. *Given a self-similar IFS $\mathcal{F} = \{f_k(x)\}_{k=1}^{m}$ on $\mathbb{R}$. We say that $\mathcal{F}$ satisfies the* ***Exponential Separation Condition (ESC)*** *if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^{\infty}$ such that*

$$\mathrm{dist}\,(f_{\overline{\imath}}, f_{\overline{\jmath}}) \geq c^{n_\ell} \text{ for all } \ell \text{ and for all } \overline{\imath}, \overline{\jmath} \in \{1, \dots, M\}^{n_\ell},\ \overline{\imath} \neq \overline{\jmath}.$$

We note that the exponential separation condition always holds when an IFS is parametrized by algebraic parameters [9]. For some of our results, it is enough to assume an even weaker separation condition on the generated self-similar iterated function systems.

Definition 2.3.2. *We say that the self-similar IFS* $\mathcal{S} = \{S_k\}_{k=1}^m$ *has* ***no exact overlapping*** *if for all* $n \geq 1$ *and all* $\mathbf{i}, \mathbf{j} \in [m]^n$ *we have*

$$S_{\mathbf{i}} \equiv S_{\mathbf{j}} \implies \mathbf{i} = \mathbf{j}.$$

Examples of IFSs for which the exponential separation condition fails but there are no exact overlappings were given by Baker [1] and Bárány, Käenmäki [2].

Theorem 2.3.3 (Hochman [9])**.** *Let* $\mathcal{F}$ *be a self-similar IFS on* $\mathbb{R}$ *which satisfies the Exponential Separation Condition (ESC). Then*

(a) *For every self-similar measure* ν *we have*

$$\dim_{\mathrm{H}} \nu = \min\{1, \dim_{\mathrm{S}} \Lambda\}.$$

(b) *Consequently,*

$$\dim_{\mathrm{H}} \Lambda = \min\{1, \dim_{\mathrm{S}} \Lambda\}.$$

The following theorem of Hochman is about the translation family of a self similar IFS. To state it we need some further notation.

Definition 2.3.4. *We consider the translation family* $\{\mathcal{F}^{\boldsymbol{\tau}}\}_{\boldsymbol{\tau} \in \mathbb{R}^M}$ *(defined in Definition 1.1.2) of a self-similar IFS* $\mathcal{F}$*. We denote the attractor of* $\mathcal{F}^{\tau}$ *by* $\Lambda^{\boldsymbol{\tau}}$*. For* $\bar{\imath}, \bar{\jmath} \in [M]^n$ *we write*

$$\Delta_{\bar{\imath},\bar{\jmath}}(\boldsymbol{\tau}) := f_{\bar{\imath}}^{\boldsymbol{\tau}}(0) - f_{\bar{\jmath}}^{\boldsymbol{\tau}}(0).$$

Let us define the exceptional set

$$E := \bigcap_{\varepsilon > 0} \left(\bigcup_{N=1}^{\infty} \bigcap_{n > N} \left(\bigcup_{\substack{\bar{\imath},\bar{\jmath} \in [M]^n \\ \bar{\imath} \neq \bar{\jmath}}} \Delta_{\bar{\imath},\bar{\jmath}}^{-1}(-\varepsilon^n, \varepsilon^n) \right) \right)$$

One can easily check that this simple claim holds:

Claim 2.3.5. *Using the notation of Definition 2.3.4, we have*

$$\left\{ \boldsymbol{\tau} \in \mathbb{R}^M : \mathcal{F}^{\boldsymbol{\tau}} \textit{ does not satisfy the ESC } \right\} = E.$$

The next theorem is a Corollary of [10, Theorem 1.10].

Theorem 2.3.6 (Multi parameter Hochman Theorem [10]). *Using the notation of Definition 2.3.4, we have*

$$\dim_{\mathrm{P}} E \leq M - 1.$$

Consequently,

$$\dim_{\mathrm{P}} \left\{ \boldsymbol{\tau} \in \mathbb{R}^M : \mathcal{F}^{\boldsymbol{\tau}} \text{ does not satisfy the ESC } \right\} \leq M - 1.$$

The consequences of the following theorem will be very important in Chapter 3.

Theorem 2.3.7 (Jordan, Rapaport[14]). *Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a self-similar IFS on $\mathbb{R}$ which satisfies the ESC. Moreover, let μ be an invariant ergodic probability measure on $\Sigma = [M]^{\mathbb{N}}$. Then*

$$\dim_{\mathrm{H}} \Pi_* \mu = \min \left\{ 1, \frac{h(\mu)}{\chi(\mu)} \right\}.$$

In Section 3.3 in Corollary 3.3.2, we prove that with the help of this result we can extend part (b) of Theorem 2.3.3 to self-similar graph-directed iterated function systems.

2.3.2 Graph-directed iterated function systems

We present here the most important notations and results related to self-similar Graph-Directed Iterated function Systems (GDIFS). In this subsection we follow the book [8] and the papers [16] and [15]. The major difference is that in the first two references the authors assume separation in between the graph-directed sets. In [15] no separation is assumed, and we follow that line.

To define the graph-directed iterated function systems we need a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We label the vertices of this graph with the numbers $\{1, 2, ..., q\}$, where $|\mathcal{V}| = q$. This $\mathcal{G}$ graph is not assumed to be simple, it might have multiple edges between the same vertices, or even loops. For an edge $e = (i, j) \in \mathcal{E}$ we write $s(e) := i$ for the source and $t(e) := j$ for the target of e. Denote with $\mathcal{E}_{i,j}$ the set of directed edges from vertex i to vertex j, and write $\mathcal{E}_{i,j}^k$ for the set of length k directed paths between i and j. Similarly, we write $\mathcal{E}^n$ for the set of all directed paths of length n in the graph. We assume that $\mathcal{G}$ is strongly connected. That is for every $i, j \in \mathcal{V}$ there is a directed path in $\mathcal{G}$ from i to j.

For all edges $e \in \mathcal{E}$ given a contracting similarity mapping $F_e : \mathbb{R} \to \mathbb{R}$. The contraction ratio is denoted by $r_e \in (-1, 1) \setminus \{0\}$. Let $e_1 \dots e_n$ be a path in $\mathcal{G}$. Then we write $F_{e_1 \dots e_n} := F_{e_1} \circ \cdots \circ F_{e_n}$. It follows from the proof of [16, Theorem 1.1] that there exists a unique family of non-empty compact sets $\Lambda_1, ..., \Lambda_q$ labeled by the elements of $\mathcal{V}$, for which

$$\Lambda_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} F_e(\Lambda_j), \quad i = 1, \dots, q. \tag{2.10}$$

We call the sets $\{\Lambda_1, ..., \Lambda_q\}$ **graph-directed sets**, and we say that $\Lambda := \bigcup\limits_{i=1}^{q} \Lambda_i$ is **the attractor of the self-similar graph-directed IFS** $\mathcal{F} = \{F_e\}_{e \in \mathcal{E}}$. We abbreviate it to **self-similar GDIFS**.

By iterating (2.10) we get

$$\Lambda_i = \bigcup_{j=1}^{q} \bigcup_{(e_1, \dots e_k) \in \mathcal{E}_{i,j}^k} F_{e_1 \dots e_k}(\Lambda_j).$$

To get the most natural guess for the dimension of Λ, for every $s \geq 0$, we define a $q \times q$ matrix with the following entries

$$\mathbf{C}^{(s)} = (c^{(s)}(i,j))_{i,j=1}^{q} \text{ and } c^{(s)}(i,j) = \begin{cases} 0, & \text{if } \mathcal{E}_{i,j} = \emptyset; \\ \sum\limits_{e \in \mathcal{E}_{i,j}} |r_e|^s, & \text{otherwise.} \end{cases} \tag{2.11}$$

The spectral radius of $\mathbf{C}^{(s)}$ is denoted by $\varrho(\mathbf{C}^{(s)})$. Mauldin and Williams [16, Theorem 2] proved that the function $s \mapsto \varrho(\mathbf{C}^{(s)})$ is strictly decreasing, continuous, greater than 1 at $s = 0$, and less than 1 if s is large enough. Therefore, the following α value is well defined.

Definition 2.3.8. *For the self-similar GDIFS $\mathcal{F} = \{f_e\}_{e \in \mathcal{E}}$ we write $\alpha = \alpha(\mathcal{F})$ for the unique number satisfying*

$$\varrho(\mathbf{C}^{(\alpha)}) = 1.$$

The relation of α to the dimension of the attractor is given by the following theorem. It appeared in [16] apart from the box dimension part, which is from [8].

Theorem 2.3.9. *Let $\mathcal{F} = \{F_e\}_{e \in E}$ be a self-similar GDIFS as above. In particular, the graph $\mathcal{G} = (\mathcal{V}, E)$ is strongly connected and let Λ be the attractor.*

(a) $\dim_{\mathrm{H}} \Lambda \leq \alpha$.

(b) *Let I_k be the convex hull of Λ_k, $\forall k \in \mathcal{V}$. If the intervals $\{I_k\}_{k=1}^{q}$ are pairwise disjoint then $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \alpha$ and $0 < \mathcal{H}^{\alpha}(\Lambda) < \infty$.*

Chapter 3

CPLIFSs with small slopes

Definition 3.0.1. *We say that $\mathcal{F}$* ***is small*** *if both of the following two requirements hold:*

(a) $\sum_{k=1}^{m} \rho_k < 1$.

(b) *Our second requirement depends on the injectivity of f_k:*

- **(i)** *If f_k is injective then we require that $\rho_k < \frac{1}{2}$.*
- **(ii)** *If f_k is not injective then we require that $\rho_k < \frac{1-\rho_{\max}}{2}$, which always holds if $\rho_{\max} < \frac{1}{3}$.*

The first assumption is required to know a priori that $\dim_{\mathrm{H}} \Lambda < 1$ in a certain uniform manner, and the later assumptions are required for a kind of transversality argument.

Along the lines of Definition 3.0.1, we define the set of small contraction vectors for a given type $\boldsymbol{\ell}$ as

$$\mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}} := \left\{ \boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}} : \mathcal{F}^{(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})} \text{ is small for all } \mathfrak{b} \in \mathfrak{B}^{\boldsymbol{\ell}} \text{ and } \boldsymbol{\tau} \in \mathbb{R}^m \right\}.$$

We will show that the following property is typical among small CPLIFSs.

Definition 3.0.2. *We say that a small CPLIFS $\mathcal{F}$ is* ***regular*** *if its attractor $\Lambda^{\mathcal{F}}$ does not contain any of the breaking points $\{b_{k,i}\}_{k \in [m], i \in [l(k)]}$.*

Regular systems can be represented with a well chosen self-similar graph-directed iterated function system (GDIFS). This property will enable us to apply notions and theorems related to self-similar GDIFSs [18].

3.1 IFSs associated to a regular CPLIFS

In this rather technical section we will complete the following steps:

(1) First we define the self-similar IFS $\mathcal{S}_{\mathcal{F}}$ for a CPLIFS $\mathcal{F}$.

(2) We assume that $\mathcal{F}$ is regular and we consider a suitable sub-system $\mathscr{S}_{\mathcal{F}}$ (called associated IFS) of the N-th iterate $\mathcal{S}_{\mathcal{F}}^N$ of the generated self-similar IFS $\mathcal{S}_{\mathcal{F}}$ for the appropriate N.

(3) We represent $\Lambda^{\mathcal{F}}$ as the attractor of a graph directed self-similar IFS whose functions are elements of $\mathscr{S}_{\mathcal{F}}$.

(4) Then we define an appropriate invariant and ergodic measure (actually a Markov measure) for the self-similar IFS $\mathscr{S}_{\mathcal{F}}$, whose support is the attractor of the previously mentioned graph directed self-similar IFS which coincides with $\Lambda_{\mathcal{F}}$ as we mentioned above.

Later we will apply the Jordan Rapaport theorem (Theorem 2.3.7) for this measure to get the dimension of $\Lambda^{\mathcal{F}}$.

Let $\mathcal{F} = \{f_k\}_{k=1}^m \in \text{CPLIFS}_{\boldsymbol{\ell}}$ for $\boldsymbol{\ell} = (l(1), \dots, l(m))$ with $L = \sum_{k=1}^{m} l(k)$. The **generated self-similar IFS** $S_{\mathcal{F}}$ consists of those similarity mappings on $\mathbb{R}$ whose graph coincides with the graph of f_k for some $k \in [m]$ on some interval of linearity $J_{k,i}$, $i \in l(k)+1$ of f_k. Hence, it is natural to parameterize the generated self-similar IFS by the $l(k)+1$ intervals of linearity $J_{k,1}, \dots, J_{k,l(k)+1}$ of the mapping f_k for all $k \in [m]$. That is

$$\mathcal{S} := \mathcal{S}_{\mathcal{F}} = \{S_{k,i}(x) = \rho_{k,i} \cdot x + t_{k,i}\}_{k \in [m], i \in [l(k)+1]}, \quad S_{k,i}|_{J_{k,i}} \equiv f_k|_{J_{k,i}}. \tag{3.1}$$

We organize the translation parts $\{t_{k,i}\}_{k \in [m], i \in [l(k)+1]}$ of the mappings of $\mathcal{S}_{\mathcal{F}}$ into a vector

$$\mathfrak{t} = \mathfrak{t}(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho}) := (t_{1,1}, \dots, t_{1,l(1)+1}, \dots, t_{m,1}, \dots, t_{m,l(m)+1}) \in \mathbb{R}^{L+m}. \tag{3.2}$$

If $b_{k,i} > 0$ for all $i \in [l(k)]$ then the formula for t_k is simple:

$$t_{k,1} = \tau_k \text{ and } t_{k,i} = \tau_k + \sum_{p=1}^{i} b_{k,p}(\rho_{k,p} - \rho_{k,p+1}). \tag{3.3}$$

Using this and an appropriate conjugation in the general case when some of $b_{k,i} \leq 0$ we obtain by simple calculation that the following statement holds:

Claim 3.1.1. *For any fixed $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}$, consider the mapping $\Phi_{\boldsymbol{\rho}} : \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \to \mathbb{R}^{L+m}$ defined by*

$$\Phi_{\boldsymbol{\rho}}(\mathfrak{b}, \boldsymbol{\tau}) := \mathfrak{t}, \tag{3.4}$$

where $\mathfrak{t} \in \mathbb{R}^{L+m}$ was defined in (3.2). Then $\Phi_{\boldsymbol{\rho}}$ is a non-singular affine transformation. Hence $\Phi_{\boldsymbol{\rho}}$ and its inverse $\Phi_{\boldsymbol{\rho}}^{-1}$ preserve Hausdorff and packing dimensions, and also preserve the sets of zero measure with respect to $\mathcal{L}^{L+m}$.

To get a more effective method of labeling the $f_k \in \mathcal{F}$ functions' intervals of linearity, and in this way the functions in $S_{\mathcal{F}}$, we introduce

$$\mathcal{A} := \{(k, i) : k \in [m] \text{ and } i \in [l(k) + 1]\} .$$

Moreover, we write $\mathcal{A}^N := \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{N}$ for $N \in \mathbb{N} \cup \{\infty\}$. That is $\mathcal{S}_{\mathcal{F}} = \{S_a\}_{a \in \mathcal{A}}$. Note that $\#\mathcal{A} = L + m$. Recall that the slope ρ_a of S_a was defined in (2.2), and we define $t_a \in \mathbb{R}$ such that

$$S_a(x) = \rho_a x + t_a.$$

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a regular CPLIFS and let $N = N(\mathcal{F}) \in \mathbb{N}$ be the smallest n such that for all $\mathbf{u} = (u_1, \dots, u_n) \in [m]^n$ the cylinder interval $I_{\mathbf{u}}^{\mathcal{F}}$ does not contain any breaking points. Then we say that $\mathcal{F}$ **is regular of order** N. The collection of regular CPLIFS of type $\boldsymbol{\ell}$ and order N is denoted by $\text{CPLIFS}_{\boldsymbol{\ell},N}$.

Given an $\mathcal{F} \in \text{CPLIFS}_{\boldsymbol{\ell},N}$. Let $\mathcal{G}_{\mathcal{F}}$ be the following directed full graph

$$\mathcal{G}_{\mathcal{F}} := (\mathcal{V}, \mathcal{E}) \text{ with } \mathcal{V} := [m]^N \text{ and } \mathcal{E} := \{(\mathbf{v}, \mathbf{u}) : \mathbf{v}, \mathbf{u} \in \mathcal{V}\} \tag{3.5}$$

Recall from Section 2.3.2 that we denote the source and the target of an edge $e \in \mathcal{E}$ by $s(e)$ and $t(e)$ respectively. Further, for a $k \in \mathbb{N}$ and $\mathbf{v}, \mathbf{u} \in \mathcal{V}$ we write $\mathcal{E}_{\mathbf{v},\mathbf{u}}^k$ for the set of length k directed paths $(e_1, \dots, e_k)$ such that $s(e_1) = \mathbf{v}$, $t(e_k) = \mathbf{u}$ and $t(e_i) = s(e_{i+1})$ for $i \in [k-1]$.

This section is organized in two steps:

(a) First we associate a self-similar IFS $\mathscr{S}_{\mathcal{F}}$ to every $\mathcal{F} \in \text{CPLIFS}_{\boldsymbol{\ell},N}$ which is the relevant subsystem of $\mathcal{S}_{\mathcal{F}}^N$.

(b) Then, we use the directed graph $\mathcal{G}_{\mathcal{F}} = (\mathcal{V}, \mathcal{E})$ to construct a graph-directed self-similar IFS $\{F_e\}_{e \in \mathcal{E}}$ which consists of the functions of $\mathscr{S}_{\mathcal{F}}$. Moreover, the attractor $\Lambda^{\mathcal{G}_{\mathcal{F}}}$ of this self-similar GDIFS coincides with $\Lambda^{\mathcal{F}}$.

3.1.1 The construction of $\mathscr{S}_{\mathcal{F}}$

Let $e = (\mathbf{v}, \mathbf{u}) \in \mathcal{E}$. For a $p \in [N]$ we consider $f_{\sigma^p \mathbf{v}} I_{\mathbf{u}}$, where σ is the left shift. That is

$$f_{\sigma^p \mathbf{v}} I_{\mathbf{u}} = \begin{cases} f_{v_{p+1} \dots v_N} I_{\mathbf{u}}, & \text{if } p \in [N-1]; \\ I_{\mathbf{u}}, & \text{if } p = N. \end{cases}$$

Clearly, $f_{\sigma^p \mathbf{v}} I_{\mathbf{u}}$ is contained in the N-cylinder $I_{v_{p+1} \dots v_N u_1 \dots u_p}$. That is $f_{\sigma^p \mathbf{v}} I_{\mathbf{u}}$ contains no breaking points of any functions from $\mathcal{F}$. Namely, $f_{\sigma^p \mathbf{v}} I_{\mathbf{u}}$ is the subset of a linearity interval of f_{v_p} for every $p \in [N]$.

In particular, there exists a unique $i(e,p) \in [l(v_p)+1]$ such that

$$f_{\sigma^p \mathbf{v}} I_{\mathbf{u}} \subset J_{v_p, i(e,p)}, \text{ for any } p \in [N] \tag{3.6}$$

Now we define a mapping $\psi = \psi_{\mathcal{F}} : \mathcal{E} \to \mathcal{A}^N$

$$\psi(e) = \mathbf{a} = (a_1, \dots, a_N), \text{ where } a_p := (v_p, i(e,p)). \tag{3.7}$$

Let $e = (\mathbf{v}, \mathbf{u}), e' = (\mathbf{v}', \mathbf{u}') \in \mathcal{E}$. It is immediate from the construction that

$$\psi(e) = \psi(e') \Longrightarrow \mathbf{v} = s(e) = s(e') = \mathbf{v}'. \tag{3.8}$$

We define $\mathscr{A}$ as the image of $\mathcal{E}$ under ψ

$$\mathscr{A} := \mathscr{A}_{\mathcal{F}} := \left\{ \mathbf{a} \in \mathcal{A}^N : \exists e \in \mathcal{E}, \mathbf{a} = \psi(e) \right\}.$$

Hence, for an $\mathbf{a} \in \mathscr{A}$ it makes sense to write

$$\psi_1^{-1}(\mathbf{a}) := \mathbf{v} \quad \text{and} \quad \mathbf{u} \in \psi_2^{-1}(\mathbf{a}), \quad \text{if } \psi(e) = \mathbf{a} \text{ and } e = (\mathbf{v}, \mathbf{u}).$$

Put

$$S_{\mathbf{a}} := S_{a_1} \circ \dots \circ S_{a_N} \text{ for } \psi(e) = \mathbf{a} = (a_1, \dots, a_N).$$

By (3.1) for any $p \in [N]$ and $a_p = (v_p, i(e,p))$

$$S_{a_p}|_{J_{v_p, i(e,p)}} \equiv S_{v_p, i(e,p)}|_{J_{v_p, i(e,p)}} \equiv f_{v_p}|_{J_{v_p, i(e,p)}}. \tag{3.9}$$

Then, using (3.9) and (3.6) together it is clear that

$$S_{\mathbf{a}}|_{I_{\mathbf{u}}} \equiv f_{\mathbf{v}}|_{I_{\mathbf{u}}}, \text{ for } \mathbf{a} = \psi(e) \text{ with } e = (\mathbf{v}, \mathbf{u}) \in \mathcal{E}.$$

With the help of alphabet $\mathscr{A}$ we define the **associated self-similar IFS**

$$\mathscr{S} := \mathscr{S}_{\mathcal{F}} := \{S_{\mathbf{a}}\}_{\mathbf{a} \in \mathscr{A}}.$$

The mapping $\psi : \mathcal{E} \to \mathscr{A}$, which we use to associate the elements of $\mathscr{A}$ to edges, is onto but not necessarily $1-1$.

3.1.2 The associated graph-directed self-similar IFS

For an $e = (\mathbf{v}, \mathbf{u}) \in \mathcal{E}$ we consider $S_{\mathbf{a}}$ for $\mathbf{a} = (a_1, \ldots, a_N) = \psi(e)$ and define the mapping $F_e : I_{\mathbf{u}} \to I_{\mathbf{v}}$ by

$$F_e(x) := S_{\mathbf{a}}|_{I_{\mathbf{u}}} = f_{\mathbf{v}}|_{I_{\mathbf{u}}}. \tag{3.10}$$

Moreover, for $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}^k_{\mathbf{v}^1, \mathbf{v}^{k+1}}$ with $e_i = (\mathbf{v}^i, \mathbf{v}^{i+1})$, $i \in [k]$ we have

$$\forall x \in I_{\mathbf{v}^{k+1}} : \quad F_{\mathbf{e}}(x) := F_{e_1} \circ \cdots \circ F_{e_k}(x) = f_{\mathbf{v}^1} \circ \cdots \circ f_{\mathbf{v}^k}(x). \tag{3.11}$$

Obviously, F_e is a similarity mapping restricted to $I_{\mathbf{u}}$ with contraction ratio

$$\rho_{\mathbf{v},\mathbf{u}} := \rho_e := \rho_{\mathbf{a}} := S'_{\mathbf{a}} \equiv \rho_{a_1} \cdots \rho_{a_N}. \tag{3.12}$$

Recall that we defined the directed full graph in (3.5). We call $\mathcal{F}^{\mathcal{G}} := \{F_e\}_{e \in \mathcal{E}}$ the **associated self-similar graph-directed IFS**. The symbolic space of $\mathcal{F}^{\mathcal{G}}$ is the set of infinite paths in the full graph $\mathcal{G}_{\mathcal{F}}$ that we denote by $\mathcal{E}_\infty$

$$\begin{aligned} \mathcal{E}_\infty &:= \{\mathfrak{p} := (e_1, e_2, \ldots) : t(e_i) = s(e_{i+1}),\ e_i \in \mathcal{E} \text{ for all } i \in \mathbb{N}\} \\ &= \left\{ \mathfrak{p} = (\underbrace{(\mathbf{v}^1, \mathbf{v}^2)}_{e_1}, \underbrace{(\mathbf{v}^2, \mathbf{v}^3)}_{e_2}, \underbrace{(\mathbf{v}^3, \mathbf{v}^4)}_{e_3}, \ldots) : \mathbf{v}^k \in \mathcal{V}, \quad \forall k \in \mathbb{N} \right\}. \end{aligned}$$

The attractor of $\mathcal{F}^{\mathcal{G}}$ is

$$\Lambda^{\mathcal{F}^{\mathcal{G}}} := \bigcup_{\mathbf{v} \in \mathcal{V}} \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{u} \in \mathcal{V}} \bigcup_{(e_1 \ldots e_k) \in \mathcal{E}^k_{\mathbf{v},\mathbf{u}}} F_{e_1 \ldots e_k} I_{\mathbf{u}}.$$

As the graph-directed sets of $\mathcal{F}^{\mathcal{G}}$ are the level N cylinder intervals of $\mathcal{F}$ and $\mathcal{G}$ is a full graph, by (3.11) we have

$$\Lambda^{\mathcal{F}} = \Lambda^{\mathcal{F}^{\mathcal{G}}}.$$

Thus $\Lambda^{\mathcal{F}}$ can be represented as a graph-directed attractor. In the next section we show how $\Lambda^{\mathcal{F}}$ relates to the associated self-similar IFS $\mathscr{S}_{\mathcal{F}}$.

3.1.3 Constructing a subshift in $\mathscr{A}^{\mathbb{N}}$

Now we show that we can consider $\Lambda^{\mathcal{F}}$ also as the projection of a subshift $\mathscr{A}^{\mathbb{N}}_{\text{Good}} \subset \mathscr{A}^{\mathbb{N}}$ (defined below) by $\Pi_{\mathscr{S}_{\mathcal{F}}}$, the natural projection corresponding to $\mathscr{S}_{\mathcal{F}}$. In order to construct $\mathscr{A}^{\mathbb{N}}_{\text{Good}}$ we define two bijections: $\vartheta : \mathcal{V}^{\mathbb{N}} \to \mathcal{E}_\infty$ and $\Psi : \mathcal{E}_\infty \to \mathscr{A}^{\mathbb{N}}_{\text{Good}}$.

Our first bijection ϑ is a very simple one:

$$\vartheta(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4, \dots) := \big((\mathbf{v}^1, \mathbf{v}^2), (\mathbf{v}^2, \mathbf{v}^3), (\mathbf{v}^3, \mathbf{v}^4), \dots\big) .$$

To define our second bijection, first recall the definition of $\psi : \mathcal{E} \to \mathscr{A}$ from (3.7). Now we apply this componentwise to define the mapping Ψ on $\mathcal{E}_\infty$ by

$$\Psi(e_1, e_2, \dots) = \big(\mathbf{a}^1, \mathbf{a}^2, \dots\big) \in \mathscr{A}^{\mathbb{N}}, \quad \mathbf{a}^k := \psi(e_k), \quad k \in \mathbb{N}.$$

Let

$$\mathscr{A}^{\mathbb{N}}_{\text{Good}} := \Psi(\mathcal{E}_\infty).$$

We claim that $\Psi : \mathcal{E}_\infty \to \mathscr{A}^{\mathbb{N}}_{\text{Good}}$ is a bijection. By definition $\psi(e_i) = \mathbf{a}^i$ for $i = 1, \dots, k$. By (3.8), this determines $e_1, \dots, e_{k-1}$ uniquely. Since this holds for all $k \in \mathbb{N}$ we obtain that Ψ is $1-1$. Clearly $\mathscr{A}^{\mathbb{N}}_{\text{Good}}$ is compact and forward invariant. That is

$$\sigma(\mathscr{A}^{\mathbb{N}}_{\text{Good}}) \subset \mathscr{A}^{\mathbb{N}}_{\text{Good}} \subset \mathscr{A}^{\mathbb{N}}. \tag{3.13}$$

That is $\mathscr{A}^{\mathbb{N}}_{\text{Good}}$ is a subshift in $\mathcal{A}^{\mathbb{N}}$.

As we defined earlier, let $\Pi_{\mathcal{F}^N} : \mathcal{V}^{\mathbb{N}} \to \mathbb{R}$ and $\Pi_{\mathscr{S}_\mathcal{F}} : \mathscr{A}^{\mathbb{N}} \to \mathbb{R}$ be the natural projections corresponding to the N-th iterate system of the CPLIFS $\mathcal{F}$ and the self-similar IFS $\mathscr{S}_\mathcal{F}$ respectively. (For the definition of the natural projection of a general IFS see Section 2.3.)

Consider the following two diagrams:

$$\begin{array}{ccc} \mathcal{V}^{\mathbb{N}} & \xrightarrow{\sigma} & \mathcal{V}^{\mathbb{N}} \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\vartheta} \\ \mathcal{E}_\infty & \xrightarrow[\sigma]{} & \mathcal{E}_\infty \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \mathscr{A}^{\mathbb{N}}_{\text{Good}} & \xrightarrow[\sigma]{} & \mathscr{A}^{\mathbb{N}}_{\text{Good}} \end{array} \qquad \begin{array}{ccc} \mathcal{V}^{\mathbb{N}} & \xrightarrow{\Psi\circ\vartheta} & \mathscr{A}^{\mathbb{N}}_{\text{Good}} \\ & {\scriptstyle\Pi_{\mathcal{F}^N}}\searrow & \downarrow{\scriptstyle\Pi_{\mathscr{S}_\mathcal{F}}} \\ & & \Lambda^{\mathcal{F}} \end{array} \tag{3.14}$$

It is obvious that the first diagram is commutative.

Claim 3.1.2. *The second diagram in* (3.14) *is commutative.*

Proof. Write $\Pi_{\mathcal{F}^\mathcal{G}}$ for the natural projection defined by the associated self-similar GDIFS $\mathcal{F}^\mathcal{G}$. Since $\Lambda^\mathcal{F} = \Lambda^{\mathcal{F}^\mathcal{G}}$, we may dissect the diagram in the following way:

$$\begin{array}{ccc} \mathcal{V}^{\mathbb{N}} & \xrightarrow{\vartheta} & \mathcal{E}_\infty \\ & {\scriptstyle\Pi_{\mathcal{F}^N}}\searrow & \downarrow{\scriptstyle\Pi_{\mathcal{F}^\mathcal{G}}} \\ & & \Lambda^{\mathcal{F}} \end{array} \qquad \begin{array}{ccc} \mathcal{E}_\infty & \xrightarrow{\Psi} & \mathscr{A}^{\mathbb{N}}_{\text{Good}} \\ \downarrow{\scriptstyle\Pi_{\mathcal{F}^\mathcal{G}}} & \swarrow{\scriptstyle\Pi_{\mathscr{S}_\mathcal{F}}} & \\ \Lambda^{\mathcal{F}^\mathcal{G}} & & \end{array} \tag{3.15}$$

It follows from the definition of ϑ and (3.11) that the left diagram in (3.15) is commutative. Using (3.10) and the definition of $\mathscr{A}^{\mathbb{N}}_{\text{Good}}$ it is easy to see that the right diagram is also commutative.

□

In particular we obtained that

$$\Lambda^{\mathcal{F}} = \Lambda^{\mathcal{F}^N} = \Pi_{\mathcal{F}^N}(\mathcal{V}^{\mathbb{N}}) = \Pi_{\mathscr{S}_{\mathcal{F}}}\left(\mathscr{A}^{\mathbb{N}}_{\text{Good}}\right). \tag{3.16}$$

3.1.4 An ergodic measure on $\Lambda^{\mathscr{S}_{\mathcal{F}}}$ supported by $\Lambda^{\mathcal{F}}$

Our aim is to define an appropriate invariant ergodic measure $\mathfrak{m}$ on $\mathscr{A}^{\mathbb{N}}$ which is supported by $\mathscr{A}^{\mathbb{N}}_{\text{Good}}$. Then we will take its push-forward measure by $\Pi_{\mathscr{S}_{\mathcal{F}}}$. This way, according to (3.16), the push-forward measure $\Pi_{\mathscr{S}_{\mathcal{F}}*}\mathfrak{m}$ is supported by $\Lambda^{\mathcal{F}}$.

For every $\beta \geq 0$ we define the $m^N \times m^N$ matrix $\mathbf{C}^{(\beta)}$ just like we did in Subsection 2.3.2. First we order the elements of $\mathcal{V}$ according to lexicographical order. This will be the order of the rows and columns of $\mathbf{C}^{(\beta)}$. Then set

$$\mathbf{C}^{(\beta)} := (|\rho_{\mathbf{v},\mathbf{u}}|^{\beta})_{(\mathbf{v},\mathbf{u})\in\mathcal{V}\times\mathcal{V}}, \tag{3.17}$$

where $\rho_{\mathbf{v},\mathbf{u}}$ was defined in (3.12). According to Definition 2.3.8, the number $\alpha = \alpha(\mathcal{F})$ is uniquely defined by $\varrho\left(\mathbf{C}^{(\alpha)}\right) = 1$.

Since $\mathbf{C}^{(\alpha)}$ is an irreducible matrix, both the left and the right eigenvectors $\underline{\mathfrak{u}} = (\mathfrak{u}_{\mathbf{v}})_{\mathbf{v}\in\mathcal{V}}$, $\underline{\mathfrak{v}} = (\mathfrak{v}_{\mathbf{v}})_{\mathbf{v}\in\mathcal{V}}$ corresponding to eigenvalue 1 can be choosen to have all positive components. That is

$$\sum_{\mathbf{v}\in\mathcal{V}} \mathfrak{u}_{\mathbf{v}} \cdot |\rho_{\mathbf{v},\mathbf{u}}|^{\alpha} = \mathfrak{u}_{\mathbf{u}}, \quad \sum_{\mathbf{u}\in\mathcal{V}} |\rho_{\mathbf{v},\mathbf{u}}|^{\alpha}\mathfrak{v}_{\mathbf{u}} = \mathfrak{v}_{\mathbf{v}}, \quad \mathfrak{u}_{\mathbf{v}}, \mathfrak{v}_{\mathbf{v}} > 0 \text{ for all } \mathbf{v} \in \mathcal{V}. \tag{3.18}$$

We normalize them in such a way that

$$\sum_{\mathbf{v}\in\mathcal{V}} \mathfrak{v}_{\mathbf{v}} = 1, \quad \sum_{\mathbf{v}\in\mathcal{V}} \mathfrak{u}_{\mathbf{v}} \cdot \mathfrak{v}_{\mathbf{v}} = 1.$$

Now we define the stochastic matrix $P = (p_{\mathbf{v},\mathbf{u}})_{\mathbf{v},\mathbf{u}\in\mathcal{V}}$ and its stationary distribution $\mathbf{p} = (p_{\mathbf{v}})_{\mathbf{v}\in\mathcal{V}}$, which corresponds to the matrix $\mathbf{C}^{(\alpha)}$. That is

$$p_{\mathbf{v},\mathbf{u}} := \frac{|\rho_{\mathbf{v},\mathbf{u}}|^{\alpha}\mathfrak{v}_{\mathbf{u}}}{\mathfrak{v}_{\mathbf{v}}}, \qquad p_{\mathbf{v}} := (\mathfrak{v}_{\mathbf{v}} \cdot \mathfrak{u}_{\mathbf{v}}), \quad \mathbf{v}, \mathbf{u} \in \mathcal{V}. \tag{3.19}$$

Clearly,

$$\mathbf{p}^T \cdot P = \mathbf{p}^T \text{ that is } \sum_{\mathbf{v}} \mathfrak{v}_{\mathbf{v}}\mathfrak{u}_{\mathbf{v}} \cdot \frac{|\rho_{\mathbf{v},\mathbf{u}}|^{\alpha}\mathfrak{v}_{\mathbf{u}}}{\mathfrak{v}_{\mathbf{v}}} = \mathfrak{v}_{\mathbf{u}}\mathfrak{u}_{\mathbf{u}}.$$

3.1.5 Auxiliary measures

Now we consider the one-sided Markov shift on $\mathcal{V}^{\mathbb{N}}$ (see [27, p. 22]) corresponding to $(\mathbf{p}, P)$. This gives us the ergodic Borel measure $\mu = \mu_{\mathcal{F}}$ on $\mathcal{V}^{\mathbb{N}}$ defined on the n-cylinders $[\mathbf{v}^1, \ldots, \mathbf{v}^n] \subset \mathcal{V}^n$ by

$$\mu\left(\left[\mathbf{v}^1, \ldots, \mathbf{v}^n\right]\right) := p_{\mathbf{v}^1} \cdot p_{\mathbf{v}^1,\mathbf{v}^2} p_{\mathbf{v}^2,\mathbf{v}^3} \cdots p_{\mathbf{v}^{n-1},\mathbf{v}^n}. \tag{3.20}$$

Then this extends to an ergodic measure on $\mathcal{V}^{\mathbb{N}}$ (see [27, Theorem 1.19]). Using that $\Psi \circ \vartheta : \mathcal{V}^{\mathbb{N}} \to \mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}$ is an isomorphism, and the first diagram in (3.14) is commutative, we get that the measure

$$\nu := (\Psi \circ \vartheta)_* (\mu)$$

is an invariant and ergodic measure on $(\mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}, \sigma)$. We have pointed out that $\mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}$ is a subshift in $\mathscr{A}^{\mathbb{N}}$. So we can extend ν from $\mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}$ to $\mathscr{A}^{\mathbb{N}}$ in an obvious way, such that after extension we still have an ergodic invariant measure. Namely, we define the measure $\mathfrak{m}$ on $\mathscr{A}^{\mathbb{N}}$ such that for a Borel set $H \subset \mathscr{A}^{\mathbb{N}}$

$$\mathfrak{m}(H) := \nu(H \cap \mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}). \tag{3.21}$$

Using that $\mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}$ is compact we obtain that the support $\mathrm{spt}(\mathfrak{m}) = \mathscr{A}^{\mathbb{N}}_{\mathrm{Good}}$. Moreover, it follows from (3.13) and [27, Theorem 1.6] that $\mathfrak{m}$ is an ergodic measure. The invariance of $\mathfrak{m}$ is obvious from the definition.

3.1.6 The entropy and Lyapunov exponent of $\mathfrak{m}$

First we estimate the measure of an n-cylinder for an arbitrary $n \in \mathbb{N}$. For an $\mathfrak{a} := (\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n, \ldots) \in \mathscr{A}^{\mathbb{N}}$ we write $\mathfrak{a}|_n := (\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n)$. Then

$$[\mathfrak{a}|_n] = \left\{\widehat{\mathfrak{a}} = (\widehat{\mathbf{a}}^1, \widehat{\mathbf{a}}^2, \ldots) \in \mathscr{A}^{\mathbb{N}} : \widehat{\mathbf{a}}^1 = \mathbf{a}^1, \ldots, \widehat{\mathbf{a}}^n = \mathbf{a}^n\right\}.$$

Lemma 3.1.3. *For every n and $\mathbf{v}^{n+1} \in \psi_2^{-1}(\mathbf{a}^n)$ we have*

$$\left[\mathbf{v}^1, \ldots, \mathbf{v}^n, \mathbf{v}^{n+1}\right] \subset (\Psi \circ \vartheta)^{-1} ([\mathfrak{a}|_n]) \subset \left[\mathbf{v}^1, \ldots, \mathbf{v}^n\right]$$

Proof. By the definition of $\Psi : \mathcal{E}_\infty \to \mathcal{A}^{\mathbb{N}}_{\mathrm{Good}}$

$$\Psi^{-1}([\mathfrak{a}|_n]) = \left\{\mathbf{e} \in \mathcal{E}_\infty : \psi(e_k) = \mathbf{a}^k, \forall k \in [n]\right\}.$$

That is every element of $\Psi^{-1}([\mathfrak{a}|_n])$ starts with $((\mathbf{v}^1, \mathbf{v}^2), \ldots, (\mathbf{v}^{n-1}, \mathbf{v}^n))$. As e_n is also an edge in $\mathcal{G}$, $\mathbf{v}^{n+1}$ must satisfy $\mathbf{v}^{n+1} \in \psi_2^{-1}(\mathbf{a}^n)$. □

It follows from Lemma 3.1.3 that

$$\mu\left(\left[\mathbf{v}^1,\dots,\mathbf{v}^n,\mathbf{v}^{n+1}\right]\right)\leq\nu\left(\left[\mathbf{a}^1,\dots,\mathbf{a}^n\right]\right)\leq\mu\left(\left[\mathbf{v}^1,\dots,\mathbf{v}^n\right]\right) \tag{3.22}$$

By substituting the formulas given in (3.19) for $p_{\mathbf{v},\mathbf{u}}$ and $p_{\mathbf{v}}$ into the formula (3.20) we obtain that there exist $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{\nu\left([\mathbf{a}^1,\dots,\mathbf{a}^n]\right)}{|\rho_{\mathbf{a}^1,\dots,\mathbf{a}^n}|^{\alpha}} \leq C_2.$$

By definition (see (2.9)) the Lyapunov exponent of $\mathfrak{m}$ is

$$\chi_{\mathfrak{m}} := -\sum_{\mathbf{a}\in\mathscr{A}} \mathfrak{m}([\mathbf{a}])\cdot\log|\rho_{\mathbf{a}}|,$$

where we defined $\rho_{\mathbf{a}}$ in (3.12). Then by the ergodicity of $\mathfrak{m}$ and by the Shannon-McMillan-Breiman Theorem we have

$$h(\mathfrak{m}) = -\lim_{n\to\infty}\frac{1}{n}\log\mathfrak{m}([\mathfrak{a}|_n]) \text{ for } \mathfrak{m}\text{-a.a. } \mathfrak{a}\in\mathscr{A}^{\mathbb{N}}. \tag{3.23}$$

Moreover,

$$\chi_{\mathfrak{m}} = -\lim_{n\to\infty}\frac{1}{n}\log|S'_{\mathfrak{a}|_n}| = -\lim_{n\to\infty}\frac{1}{n}\log|\rho_{\mathfrak{a}|_n}| \text{ for } \mathfrak{m}\text{-a.a. } \mathfrak{a}\in\mathscr{A}^{\mathbb{N}}. \tag{3.24}$$

Putting together (3.24),(3.23) and (3.22) we get

$$\frac{h(\mathfrak{m})}{\chi_{\mathfrak{m}}} = \alpha. \tag{3.25}$$

3.2 Dimension results for small CPLIFSs

The main theorem of this section is the following.

Theorem 3.2.1. *Let $\mathcal{F}=\{f_k\}_{k=1}^m$ be a* $\dim_{\mathrm{P}}$*-typical small CPLIFS. Then*

$$\dim_{\mathrm{H}}\Lambda^{\mathcal{F}} = \dim_{\mathrm{B}}\Lambda^{\mathcal{F}} = s_{\mathcal{F}}. \tag{3.26}$$

Further, there exists an invariant ergodic measure ν on the symbolic space with push-forward $\mu=\nu\circ\left(\Pi_{\mathcal{F}}^{-1}\right)$ supported on $\Lambda^{\mathcal{F}}$ such that

$$\dim_{\mathrm{H}}\Lambda^{\mathcal{F}} = \dim_{\mathrm{H}}\mu. \tag{3.27}$$

In Section 4, we will prove that (3.26) holds for any CPLIFS typically, without restrictions on the slopes. However, (3.27) may not hold in general, and we only prove it for regular CPLIFS. To prove our main result we verify the following theorem and proposition:

Theorem 3.2.2. *Let $\mathcal{F}$ be a regular CPLIFS for which the generated self-similar IFS satisfies the Exponential Separation Condition (ESC). Then*

$$\dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} = s_{\mathcal{F}}.$$

To prove this, we use the multi-parameter Hochman Theorem [10, Theorem 1.10] and a recent result of Jordan and Rapaport [14, Theorem 1.1].

Proposition 3.2.3. *A* $\dim_{\mathrm{P}}$*-typical small CPLIFS is regular.*

Together with [8, Theorem 3.2], this proposition implies the following corollary.

Corollary 3.2.4. *For a* $\dim_{\mathrm{P}}$*-typical small CPLIFS* $\mathcal{F}$ *let* $s := \dim_{\mathrm{H}} \Lambda^{\mathcal{F}}$ *be the Hausdorff dimension of its attractor. Further, assume that all functions in* $\mathcal{F}$ *are injective. Then we have*

$$s = \dim_{\mathrm{H}} \Lambda^{\mathcal{F}} = \dim_{\mathrm{B}} \Lambda^{\mathcal{F}}.$$

Further, $\mathcal{H}^s(\Lambda^{\mathcal{F}}) < \infty$*, where* $\mathcal{H}^s$ *denotes the* s*-dimensional Hausdorff measure.*

3.2.1 Proofs assuming Proposition 3.2.3

This section contains the proofs of our main results, under the assumption that Proposition 3.2.3 holds. Since the proof of Proposition 3.2.3 is quite lengthy, we separate it in a later subsection for didactic reasons. First, we prove some general properties of regular CPLIFSs.

Definition 3.2.5. *Let* $\mathcal{F}$ *be a CPLIFS. We say that* $\mathcal{F}$ *satisfies the* ***Bounded Distortion Property (BDP)*** *if there exist* $0 < C_1 < C_2$ *such that*

$$\forall n,\ \forall |\mathbf{j}| = n,\ \forall x, y \in I: \quad C_1 \leq \frac{|f'_{\mathbf{j}}(x)|}{|f'_{\mathbf{j}}(y)|} \leq C_2.$$

It is not true in general that a CPLIFS satisfies the BDP. For example, if a function's fixed point coincides with one of its breaking points, then this condition trivially fails. However, in the regular case the bounded distortion property does hold.

Lemma 3.2.6. *Let* $\mathcal{F}$ *be a regular CPLIFS of order* N*. Then* $\mathcal{F}$ *satisfies the bounded distortion property.*

Proof. As there are finitely many words of length at most N, the statement trivially holds for $n < N$. Let $\mathbf{j} = j_1 \dots j_n$ with $n > N$. Let $x, y \in I$ be arbitrary numbers. First we investigate the derivative of $f_{\mathbf{j}}$ at a given $x \in I$.

$$f'_{j_1,\dots,j_n}(x) = f'_{j_1}(f_{\sigma\mathbf{j}}(x)) \cdots f'_{j_{n-N}}(f_{\sigma^{n-N}\mathbf{j}}(x)) f'_{\sigma^{n-N}\mathbf{j}}(x). \tag{3.28}$$

We said that N is the order of our CPLIFS. It means that the functions of $\mathcal{F}$ have constant slopes over all the N cylinders. Hence $f'_k(\widehat{x}) = f'_k(\widehat{y})$ if $\widehat{x}, \widehat{y}$ are elements of the same cylinder interval of level N.

For $1 \le l \le n - N$ the words $\sigma^l \mathbf{j}$ have length at least N. It implies that $f_{\sigma^l\mathbf{j}}(I)$ is a subset of a cylinder interval of level N. Thus $f_{\sigma^l\mathbf{j}}(x)$ and $f_{\sigma^l\mathbf{j}}(y)$ are contained in the same cylinder interval of level N for $1 \le l \le n - N$.

We conclude that for $1 \le l \le n - N$

$$f'_k(f_{\sigma^l\mathbf{j}}(x)) = f'_k(f_{\sigma^l\mathbf{j}}(y)), \tag{3.29}$$

for any $k \in [m]$.

After substituting (3.29) into (3.28) we obtain that

$$C_1 := \frac{\rho_{\min}^N}{\rho_{\max}} \le \frac{|f'_{j_1,\dots,j_n}(x)|}{|f'_{j_1,\dots,j_n}(y)|} = \frac{|f'_{\sigma^{n-N}\mathbf{j}}(x)|}{|f'_{\sigma^{n-N}\mathbf{j}}(x)|} \le \frac{\rho_{\max}}{\rho_{\min}^N} =: C_2, \tag{3.30}$$

where ρ_{min} and ρ_{max} were defined in (2.3). □

If we assume Proposition 3.2.3, we can easily prove Corollary 3.2.4 with the help of the following theorem.

Theorem 3.2.7 (Theorem 3.2 from [8]). *Let $E \subset \mathbb{R}^n$ be a non-empty compact subset, and let $a > 0$ and $r_0 > 0$. Suppose that for every closed ball B with center in E and radius $r < r_0$ there is a mapping $g : E \to E \cap B$ satisfying*

$$ar|x - y| \le |g(x) - g(y)| \text{ for all } x, y \in E.$$

Then, writing $s = \dim_H E$, we have that

$$\mathcal{H}^s(E) \le 4^s a^{-s} < \infty \text{ and } \underline{\dim}_B(E) = \overline{\dim}_B(E) = s.$$

Proof of Corollary 3.2.4 assuming Proposition 3.2.3. It is enough to prove our statement for a small regular CPLIFS $\mathcal{F} = \{f_i\}_{i=1}^m$, in accordance with Proposition 3.2.3. Let N be the order of $\mathcal{F}$.

We proceed by applying Theorem 3.2.7 to $\Lambda^{\mathcal{F}}$. Without loss of generality we assume that $|\Lambda^{\mathcal{F}}| = 1$. Let $r_0 := 1$ and $a = \frac{\rho_{min}}{2C_2}$, where $C_2 = \frac{\rho_{\max}}{\rho_{\min}^N}$. Fix an arbitrary $0 < r < r_0$ and $x \in \Lambda^{\mathcal{F}}$, and consider the interval $D = [x - r, x + r]$.

We choose $(i_1, \dots, i_n) \in [m]^n$ such that $x \in \Lambda_{i_1,\dots,i_n} \subset D$ but $\Lambda_{i_1,\dots,i_{n-1}} \not\subset D$. Then $\frac{r}{2} < |\Lambda_{i_1,\dots,i_{n-1}}|$.

Let $I^{\mathcal{F}}$ be the interval we defined in (1.2). There exists $x_1 \in I^{\mathcal{F}}$ for which

$$\frac{r}{2} < |\Lambda_{i_1,\dots,i_{n-1}}| \le |f'_{i_1,\dots,i_{n-1}}(x_1)|. \tag{3.31}$$

According to Lemma 3.2.6 $\mathcal{F}$ satisfies the bounded distortion property. Namely, (3.30) implies that for every $x_2 \in I^{\mathcal{F}}$ we have

$$\begin{aligned} |f'_{i_1,\dots,i_{n-1}}(x_1)| &\le C_2 |f'_{i_1,\dots,i_{n-1}}(f_n(x_2))| \\ &\le C_2 \underbrace{|f'_{i_1,\dots,i_{n-1}}(f_{i_n}(x_2))| \cdot |f'_{i_n}(x_2)|}_{|f'_{i_1,\dots,i_n}(x_2)|} \frac{1}{\rho_{\min}}, \end{aligned} \tag{3.32}$$

where in the last inequality we used that $\rho_{\min}$ is the smallest slope in the system in absolute value. Together (3.31) and (3.32) gives that for all $x_2 \in I^{\mathcal{F}}$

$$|f'_{i_1,\dots,i_n}(x_2)| > \frac{\rho_{\min}}{2C_2} r = a \cdot r. \tag{3.33}$$

All the functions in $\mathcal{F}$ are continuous and piecewise linear, hence for any $y, z \in \Lambda^{\mathcal{F}}$, $y < z$ there exists a $\xi \in (y, z)$ such that

$$|f_{i_1,\dots,i_n}(z) - f_{i_1,\dots,i_n}(y)| \ge |f'_{i_1,\dots,i_n}(\xi)| \cdot (z - y) \ge a \cdot r(z - y),$$

where we used that (3.33) applies for any element of $I^{\mathcal{F}}$.

Since $f_{i_1,\dots,i_n} \Lambda^{\mathcal{F}} \subset \Lambda^{\mathcal{F}} \cap D$, all conditions of Theorem 3.2.7 hold which completes the proof of our theorem. □

Claim 3.2.8. *Let $\mathcal{F}$ be a regular CPLIFS. Then*

$$\Phi(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\dots,i_n} |I_{i_1\dots i_n}|^s.$$

That is the limit exists in (1.5).

Proof. Since $\mathcal{F}$ is regular, it satisfies the BDP according to Lemma 3.2.6. It follows that for every n there is a suitably big $k = k(n)$ with $kN \ge n - N$ and constants $D_1 = D_1(N), D_2 = D_2(N)$ such that

$$\forall (i_1, i_2, \dots) \in [m]^{\mathbb{N}} : D_1 \left| I_{i_1,\dots,i_{kN}} \right| \le \left| I_{i_1,\dots,i_n} \right| \le D_2 \left| I_{i_1,\dots,i_{kN}} \right|.$$

Using this we obtain

$$
\begin{aligned}
\Phi(s) &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\dots,i_n} |I_{i_1,\dots,i_n}|^s \qquad (3.34)\\
&= \limsup_{k\to\infty} \frac{1}{kN} \log \sum_{i_1,\dots,i_{kN}} |I_{i_1,\dots,i_{kN}}|^s \\
&= \limsup_{k\to\infty} \frac{1}{kN} \log \sum_{i_1,\dots,i_{kN}} \left| f_{i_1,\dots,i_{(k-1)N}} \left(I_{i_{(k-1)N+1},\dots,i_{kN}} \right) \right|^s \\
&= \limsup_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k\in\mathcal{V}} \left| F_{\mathbf{v}^1,\mathbf{v}^2} \circ \cdots \circ F_{\mathbf{v}^{k-1},\mathbf{v}^k} \left(I_{\mathbf{v}^k} \right) \right|^s \\
&= \limsup_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k\in\mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}|^s \cdots |\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^s \, |(I_{\mathbf{v}^k})|^s \\
&= \limsup_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k\in\mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}|^s \cdots |\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^s,
\end{aligned}
$$

where in the last equality we used that the length of any cylinder interval of level N can be easily bounded by suitable constants.

The series $c_k := \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k \in,\mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}|^s \cdots |\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^s$ is subadditive, and hence by Fekete's lemma the limit $\lim_{k\to\infty} \frac{c_k}{k}$ exists. Applying this fact to (3.34) yields

$$
\Phi(s) = \lim_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k\in\mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}|^s \cdots |\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^s. \qquad (3.35)
$$

Together (3.35) and (3.34) imply that

$$
\Phi(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\dots,i_n} |I_{i_1,\dots,i_n}|^s .
$$

□

Lemma 3.2.9. *Let $\mathcal{F}$ be a regular CPLIFS. Then we have*

$$
\alpha(\mathcal{F}) = s_{\mathcal{F}}.
$$

Proof. We define

$$
\begin{aligned}
\overline{Q} &:= \max_{i_1,\dots,i_N} |I_{i_1,\dots,i_N}|, \quad \underline{Q} := \min_{i_1,\dots,i_N} |I_{i_1,\dots,i_N}|, \qquad (3.36)\\
\overline{q} &:= \max_{\mathbf{u},\mathbf{v}\in[m]^N} \mathfrak{u}_{\mathbf{u}} \mathfrak{v}_{\mathbf{v}}, \quad \underline{q} := \min_{\mathbf{u},\mathbf{v}\in[m]^N} \mathfrak{u}_{\mathbf{u}} \mathfrak{v}_{\mathbf{v}}.
\end{aligned}
$$

Recall that $\mathfrak{u}, \mathfrak{v}$ are the left and right eigenvectors of $\mathbf{C}^{(\alpha)}$ for the eigenvalue 1 satisfying (3.18), and the matrix $\mathbf{C}^{(\alpha)}$ was defined in (3.17).

By Claim 3.2.8 and its proof we have

$$\Phi(s) = \lim_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k \in \mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}|^s \cdots |\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^s \, |(I_{\mathbf{v}^k})|^s .$$

Observe that by the definition of measure μ we have

$$\underline{q}|\rho_{\mathbf{v}^1,\mathbf{v}^2}\cdots\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^\alpha \le \mu([\mathbf{v}^1,\dots,\mathbf{v}^k]) \le \overline{q}|\rho_{\mathbf{v}^1,\mathbf{v}^2}\cdots\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^\alpha, \tag{3.37}$$

Now we substitute $s = \alpha (= \alpha_{\mathcal{F}})$ into (3.34), and use the bounds of (3.37) and the constants introduced in (3.36) to obtain

$$\begin{aligned}\Phi(\alpha) &\le \lim_{k\to\infty} \frac{1}{kN} \log \sum_{\mathbf{v}^1,\dots,\mathbf{v}^k \in \mathcal{V}} |\rho_{\mathbf{v}^1,\mathbf{v}^2}\cdots\rho_{\mathbf{v}^{k-1},\mathbf{v}^k}|^\alpha \overline{Q}^\alpha \\ &\le \lim_{k\to\infty} \frac{1}{kN} \frac{\overline{Q}^\alpha}{\underline{q}} \log \underbrace{\sum_{\mathbf{v}^1,\dots\mathbf{v}^k \in \mathcal{V}} \mu([\mathbf{v}^1,\dots\mathbf{v}^k])}_{=1} = 0,\end{aligned}$$

since μ is a probability measure. Note that it only holds for $s = \alpha(\mathcal{F})$, otherwise μ would fail to be a probability measure. Similar calculations show that 0 is also a lower bound.

$$\Phi(\alpha) \ge \lim_{k\to\infty} \frac{1}{kN} \frac{\underline{Q}^\alpha}{\overline{q}} \log \underbrace{\sum_{\mathbf{v}^1,\dots\mathbf{v}^k \in \mathcal{V}} \mu([\mathbf{v}^1,\dots\mathbf{v}^k])}_{=1} = 0.$$

As $s_{\mathcal{F}}$ is the unique zero of $\Phi(s)$, we just proved that $s_{\mathcal{F}} = \alpha$. □

Proof of Theorem 3.2.2. We fix an arbitrary regular CPLIFS $\mathcal{F}$ of order N, for which the generated self-similar IFS $\mathcal{S}_{\mathcal{F}}$ satisfies the ESC. Let $\mathscr{S}_{\mathcal{F}} := \{S_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its associated self-similar system as defined in Section 3.1. Since the ESC holds for $\mathcal{S}_{\mathcal{F}}$, it also holds for $\mathscr{S}_{\mathcal{F}}$, as it is a subsystem of $\mathcal{S}_{\mathcal{F}}^N$ (see [3]).

Recall that the measure $\mathfrak{m}$ defined in (3.21) satisfies

$$\mathrm{spt}(\Pi_{\mathscr{S}_{\mathcal{F}}*}\mathfrak{m}) = \Lambda^{\mathcal{F}}.$$

We also know that $h(\mathfrak{m})/\chi(\mathfrak{m}) = \alpha$ by (3.25). Now we can apply the Jordan Rapaport Theorem (Theorem 2.3.7) to obtain

$$\min\{1,\alpha\} \le \dim_{\mathrm{H}} \Lambda^{\mathcal{F}}.$$

Applying Lemma 3.2.9 yields

$$\min\{1, s_{\mathcal{F}}\} \leq \dim_{\mathrm{H}} \Lambda^{\mathcal{F}} \leq \dim_{\mathrm{B}} \Lambda^{\mathcal{F}} \leq \min\{1, s_{\mathcal{F}}\},$$

where the last inequality comes from Corollary 1.0.1. □

Proof of Theorem 3.2.1 assuming Proposition 3.2.3. Fix an arbitrary type $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$. Let us define the exceptional sets

$$E_{\boldsymbol{\rho}} := \left\{ (\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m : \dim_{\mathrm{H}} \Lambda^{(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})} \neq s_{(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})} \right\},$$
$$E_{\boldsymbol{\rho}, \mathrm{ESC}} := \left\{ (\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m : \mathscr{S}_{\mathcal{F}(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})} \text{ does not satisfy ESC} \right\},$$
$$E_{\boldsymbol{\rho}, \text{irregular}} := \left\{ (\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m : \mathcal{F}^{(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})} \text{ is not regular } \right\}. \tag{3.38}$$

Using this terminology Theorem 3.2.2 asserts that

$$E_{\boldsymbol{\rho}} \subset E_{\boldsymbol{\rho}, \mathrm{ESC}} \cup E_{\boldsymbol{\rho}, \text{irregular}}. \tag{3.39}$$

Recall that the mapping $\Phi_{\boldsymbol{\rho}}$ (defined in (3.4)) associates each $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ to a vector $\mathbf{t} \in \mathbb{R}^{L+m}$ uniquely. $\Phi_{\boldsymbol{\rho}}(\mathfrak{b}, \boldsymbol{\tau})$ is the translation vector of $\mathscr{S}_{\mathcal{F}(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})}$. Hence, by Hochmann's theorem (Theorem 2.3.6)

$$\dim_{\mathrm{P}} \Phi_{\boldsymbol{\rho}}(E_{\boldsymbol{\rho}, \mathrm{ESC}}) < L + m.$$

According to Claim 3.1.1, $\Phi_{\boldsymbol{\rho}}$ preserves Packing dimension, and thus

$$\dim_{\mathrm{P}} E_{\boldsymbol{\rho}, \mathrm{ESC}} < L + m.$$

It follows from Proposition 3.2.3 that

$$\dim_{\mathrm{H}} E_{\boldsymbol{\rho}, \text{irregular}} < L + m.$$

Combining (3.39) with the last two inequalities completes the proof of the first half of the Theorem.

To prove the second half, let $\mathscr{S}_{\mathcal{F}}$ be the associated self-similar IFS of $\mathcal{F}$, and let $\mathfrak{m}$ be the measure defined in (3.21). Then, according to (3.25) and Lemma 3.2.9, the invariant measure $\Pi_{\mathscr{S}_{\mathcal{F}}*}\mathfrak{m}$ satisfies (3.27). □

3.2.2 The proof of Proposition 3.2.3

Our aim in this section is to verify Proposition 3.2.3. That is we will prove that the attractor $\Lambda^{\mathcal{F}}$ of a $\dim_{\mathrm{P}}$-typical small CPLIFS $\mathcal{F}$ does not contain any breaking points. More precisely, we prove that for every $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$

$$\dim_{\mathrm{P}} (E_{\boldsymbol{\rho}, \text{irregular}}) \leq L + m - 1 + s_*, \tag{3.40}$$

where $E_{\boldsymbol{\rho}, \text{irregular}}$ was defined in (3.38) and $0 < s_* < 1$ is defined by

$$\sum_{k=1}^{m} \rho_k^{s_*} = 1.$$

Proof of Proposition 3.2.3. By symmetry, it is enough to prove that

$$b_{1,1} \notin \Lambda^{\mathcal{F}}, \qquad \text{for a } \dim_{\mathrm{P}}\text{-typical small CPLIFS } \mathcal{F}.$$

Fix an arbitrary type $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and a $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$. We always assume that $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$, and write $\Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$ for the attractor of the CPLIFS $\mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$. As usual, we write $L := \sum_{k=1}^{m} l(k)$.

For arbitrary $U < V$, let

$$E_{U,V} := \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in \left(\mathfrak{B}^{\boldsymbol{\ell}} \cap (U,V)^L\right) \times (U,V)^m : b_{1,1} \in \Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \subset (U,V)\right\}.$$

To show that (3.40) holds, it is enough to prove that for all $U < V$ we have

$$\dim_{\mathrm{P}} E_{U,V} \leq L + m - 1 + s_*. \tag{3.41}$$

From now on we fix $U < V$ and write

$$\Delta := \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in \left(\mathfrak{B}^{\boldsymbol{\ell}} \cap (U,V)^L\right) \times (U,V)^m : \Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \subset (U,V)\right\}.$$

Let D be an arbitrary closed cube contained in Δ:

$$D = D_1 \times D_2, \text{ where } D_1 \subset \mathbb{R} \text{ and } D_2 \subset \mathbb{R}^{L+m-1}.$$

To verify (3.41) we need to prove that

$$\dim_{\mathrm{P}}(E_{U,V} \cap D) \leq L + m - 1 + s_*. \tag{3.42}$$

For an $r > 0$ we define a kind of symbolic Moran cover

$$\mathcal{M}_r := \left\{(k_1, \dots, k_n) \in \Sigma^* : |\rho_{k_1 \dots k_n}| \leq r < |\rho_{k_1 \dots k_{n-1}}|\right\}. \tag{3.43}$$

By a standard argument it is easy to see that

$$\frac{1}{r^{s_*}} \leq \#\mathcal{M}_r < \frac{1}{\rho_{\min}^{s_*} r^{s_*}}. \tag{3.44}$$

Clearly, for all $r > 0$ we have

$$E_{U,V} \cap D = \bigcup_{(k_1 \dots k_n) \in \mathcal{M}_r} \widetilde{K}_{k_1,\dots,k_n}, \tag{3.45}$$

where $\widetilde{K}_{k_1,\dots,k_n} := \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in D : b_{1,1} \in \Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}_{k_1 \dots k_n}\right\}$. Let $\widetilde{N}_{k_1 \dots k_n}(r)$ be the minimal number of r-mesh cubes required to cover $\widetilde{K}_{k_1,\dots,k_n}$. Using (3.44), it is enough to prove that there exists a constant C_1 which may depend on D but independent of r and $(k_1,\dots,k_n)\in\mathcal{M}_r$ such that

$$\widetilde{N}_{k_1\dots k_n}(r)\le\frac{C_1}{r^{L+m-1}},\quad \forall r>0,(k_1,\dots,k_n)\in\mathcal{M}_r. \tag{3.46}$$

Namely, putting together this and (3.43) we get from (3.45) that

$$\overline{\dim}_{\mathrm{B}}\left(E_{U,V}\cap D\right)\le L+m-1+s_*$$

for all cubes $D\subset\Delta$, which implies (3.42). Using that Δ is the countable union of such cubes we obtain (3.41).

Without loss of generality we may assume that $U=1$. Now we fix an $r>0$ so small that for every $(\mathfrak{b},\boldsymbol{\tau})\in D$ and $(k_1,\dots,k_n)\in\mathcal{M}_r$ we have

$$f_{k_1\dots k_n}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}([0,V+1])\subset\left(\frac{1}{2},V+\frac{1}{2}\right)\text{ and }|\rho_{k_1\dots k_n}|<\min\left\{|D_1|,\frac{1}{6(V-1)}\right\}.$$

We also fix an arbitrary $(k_1,\dots,k_n)\in\mathcal{M}_r$. Now we verify (3.46). For the duration of this proof we introduce the following notation

$$\boldsymbol{\eta}:=(b_{1,2},\dots,b_{1,l(1)},\dots,b_{m,1},\dots,b_{m,l(m)},\tau_1,\dots,\tau_m)\in D_2.$$

That is, instead of $(\mathfrak{b},\boldsymbol{\tau})$, from now on we write $(b_{1,1},\boldsymbol{\eta})\in D=D_1\times D_2$. Let $B_k^{(b_{1,1},\boldsymbol{\eta})}$ be the set of breaking points of the function $f_k^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}(\cdot)$ and we write $B_{k_1\dots k_n}^{(b_{1,1},\boldsymbol{\eta})}$ for the set of breaking points of $f_{k_1\dots k_n}^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}(\cdot)$. Then

$$B_{k_1\dots k_n}^{(b_{1,1},\boldsymbol{\eta})}=B_{k_n}^{(b_{1,1},\boldsymbol{\eta})}\cup\bigcup_{p=2}^{n}f_{k_n}^{-1}\circ\dots\circ f_{k_p}^{-1}B_{k_{p-1}}^{(b_{1,1},\boldsymbol{\eta})}.$$

The complement $\left(B_{k_1\dots k_n}^{(b_{1,1},\boldsymbol{\eta})}\right)^c$ is the union for all $(i_1,\dots,i_n)\in[l(k_1)+1]\times\dots\times[l(k_n)+1]$ of the open intervals

$$\begin{aligned}J_{(k_1,i_1)\dots(k_n,i_n)}&=J_{k_n,i_n}\cap S_{k_n,i_n}^{-1}(J_{k_{n-1},i_{n-1}})\cap\dots\cap S_{k_n,i_n}^{-1}\circ\dots\circ S_{k_2,i_2}^{-1}(J_{k_1,i_1})\\&=\left\{x\in J_{k_n,i_n}:S_{k_n,i_n}(x)\in J_{k_{n-1},i_{n-1}},\dots,S_{(k_2,i_2)\dots(S_{k_n,i_n})}(x)\in J_{k_1,i_1}\right\}.\end{aligned}$$

Let $\overline{J}_{(k_1,i_1)\dots(k_n,i_n)}$ be the closure of $J_{(k_1,i_1)\dots(k_n,i_n)}$, then by the formulas (3.1) and (3.3) we obtain that for all $x\in\overline{J}_{(k_1,i_1)\dots(k_n,i_n)}$ we have

$$\begin{aligned}&f_{k_1\dots k_n}^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}(x)=\widetilde{\rho}_n\cdot x\\&+\sum_{j=1}^{n}\widetilde{\rho}_{j-1}\underbrace{\left(\tau_{k_j}+b_{k_j,1}(\rho_{k_j,1}-\rho_{k_j,2})+\dots+b_{k_j,i_j}(\rho_{k_j,i_j}-\rho_{k_j,i_j+1})\right)}_{t_{k_j,i_j}},\end{aligned}\tag{3.47}$$

where $\widetilde{\rho}_q := \prod_{z=1}^{q} \rho_{k_z,i_z}$. Using 3.47 and that $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}_{\text{small}}$, a simple calculation shows that $f^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}_{k_1\dots k_n}$ always has the following properties.

Claim 3.2.10. *For a fixed x and $k_1, \dots, k_n$ the following hold.*

(a) *The function $f^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}_{k_1\dots k_n}(x)$ is piecewise linear in all of its variables τ_k and $b_{p,q}$. So, if we fix all variables but one, then the partial derivative, against the variable which is not fixed, exists at all but finitely many points.*

The next two assertions are meant in the sense that the estimates hold where the partial derivatives exist.

(b) *For every $p \in [m]$ and $q \in [l(p)]$ we have*

$$\left| \frac{\partial f^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}_{k_1\dots k_n}(x)}{\partial b_{p,q}} \right| < \gamma := \sum_{k=1}^{\infty} \rho^k_{\max} < 1. \tag{3.48}$$

(c) *For every $k \in [m]$*

$$\left| \frac{\partial f^{(b_{1,1},\boldsymbol{\eta},\boldsymbol{\rho})}_{k_1\dots k_n}(x)}{\partial \tau_k} \right| < 2. \tag{3.49}$$

Clearly,

$$\Lambda^{(b_{1,1},\boldsymbol{\eta})}_{k_1\dots k_n} \subset I^{(b_{1,1},\boldsymbol{\eta})} := \left(f^{(b_{1,1},\boldsymbol{\eta})}_{k_1\dots k_n}(1) - (V-1)\rho_{k_1\dots k_n}, f^{(b_{1,1},\boldsymbol{\eta})}_{k_1\dots k_n}(1) + (V-1)\rho_{k_1\dots k_n} \right),$$

and we also have

$$\widetilde{K}_{k_1\dots k_n} \subset K_{k_1\dots k_n} := \left\{ (b_{1,1}, \boldsymbol{\eta}) \in D : b_{1,1} \in I^{(b_{1,1},\boldsymbol{\eta})} \right\}. \tag{3.50}$$

Like above, we write $N_{k_1\dots k_n}(r)$ for the minimal number of r-mesh cubes required to cover the set $K_{k_1\dots k_n}$. By (3.50) $\widetilde{N}_{k_1\dots k_n}(r) \le N_{k_1\dots k_n}(r)$. Hence to verify (3.46) we only need to prove that

Lemma 3.2.11. *There exists a constant C_1 which is independent of r and $(k_1, \dots, k_n) \in \mathcal{M}_r$ such that*

$$N_{k_1\dots k_n}(r) \le \frac{C_1}{r^{L+m-1}}, \quad \forall r > 0, \ (k_1, \dots, k_n) \in \mathcal{M}_r.$$

Proof of Lemma 3.2.11. Fix $r > 0$ and $(k_1, \ldots, k_n) \in \mathcal{M}_r$. We define the mapping $G : D \to \left(\frac{1}{3}, V + \frac{2}{3}\right)$,

$$G(b_{1,1}, \boldsymbol{\eta}) := f_{k_1 \ldots k_n}^{(b_{1,1}, \boldsymbol{\eta})}(1) - (V-1)\rho_{k_1 \ldots k_n}.$$

Observe that (3.48) gives

$$\left| \frac{\partial}{\partial b_{1,1}} G(b_{1,1}, \boldsymbol{\eta}) \right| < \gamma \quad \text{for all } \boldsymbol{\eta} \in D_2. \tag{3.51}$$

Then for $w := 2(V-1)\rho_{k_1 \ldots k_n}$ we have

$$K_{k_1 \ldots k_n} = \left\{ (b_{1,1}, \boldsymbol{\eta}) \in D : b_{1,1} - w < G(b_{1,1}, \boldsymbol{\eta}) < b_{1,1} \right\}.$$

We introduce the stripe $Z := \{(x, y) \in D_1 \times \mathbb{R} : x - w < y < x\}$ and for an $\boldsymbol{\eta} \in D_2$ we write

$$K_{k_1 \ldots k_n}^{\boldsymbol{\eta}} := \left\{ b_{1,1} \in D_1 : (b_{1,1}, G(b_{1,1}, \boldsymbol{\eta})) \in Z \right\}. \tag{3.52}$$

It follows from (3.48) that for all $\boldsymbol{\eta} \in D_2$ we have

$$|G(b_{1,1}, \boldsymbol{\eta}) - G(b_{1,1} + u, \boldsymbol{\eta})| < \gamma |u|, \qquad b_{1,1}, b_{1,1} + u \in D_1, \boldsymbol{\eta} \in D_2.$$

This immediately implies that

Claim 3.2.12. *For every x we have*

(a) $\Big(G(b_{1,1}, \boldsymbol{\eta}) \geq b_{1,1} - x \Big) \Longrightarrow \Big(G(b_{1,1} - u, \boldsymbol{\eta}) > b_{1,1} - u - x \Big),$
if $b_{1,1}, b_{1,1} - u \in D_1$,

(b) $\Big(G(b_{1,1}, \boldsymbol{\eta}) \leq b_{1,1} - x \Big) \Longrightarrow \Big(G(b_{1,1} + u, \boldsymbol{\eta}) < b_{1,1} + u - x \Big),$
if $b_{1,1}, b_{1,1} + u \in D_1$.

Let us write d_l and d_r for the left and right endpoints of D_1 respectively. We define a projective mapping $\text{Proj}_2 : \mathcal{P}(\mathbb{R}^{L+m}) \to \mathcal{P}(\mathbb{R}^{L+m-1})$

$$\forall A \subset \mathbb{R}^{L+m} : \ \text{Proj}_2(A) := \left\{ \boldsymbol{\eta} \in \mathbb{R}^{L+m-1} : \ \exists b_{1,1} \text{ such that } (b_{1,1}, \boldsymbol{\eta}) \in A \right\}.$$

Claim 3.2.12 implies that

$$\boldsymbol{\eta} \in \text{Proj}_2(K_{k_1 \ldots k_n}) \Longleftrightarrow d_l - w < G(d_l, \boldsymbol{\eta}) \quad \& \quad G(d_r, \boldsymbol{\eta}) < d_r.$$

That is

$$\boldsymbol{\eta} \notin \text{Proj}_2(K_{k_1 \ldots k_n}) \Longleftrightarrow G(d_l, \boldsymbol{\eta}) \leq d_l - w \text{ or } G(d_r, \boldsymbol{\eta}) \geq d_r.$$

Assume that $d_l < G(d_l, \boldsymbol{\eta})$ and $G(d_r, \boldsymbol{\eta}) < d_r$ holds. Then there is a fixed point of the function $G(\cdot, \boldsymbol{\eta})$. Let us denote it by $\mathrm{Fix}(G(\cdot, \boldsymbol{\eta}))$. Now we define the function $q : D_2 \to [d_l, d_r]$ as follows:

$$q(\boldsymbol{\eta}) := \begin{cases} \mathrm{Fix}(G(\cdot, \boldsymbol{\eta})), & \text{if } d_l < G(d_l, \boldsymbol{\eta}) \text{ and } G(d_r, \boldsymbol{\eta}) < d_r; \\ d_l, & \text{if } G(d_l, \boldsymbol{\eta}) \leq d_l; \\ d_r, & \text{if } G(d_r, \boldsymbol{\eta}) \geq d_r. \end{cases} \tag{3.53}$$

By (3.51), this function is well defined, as $\gamma < 1$.

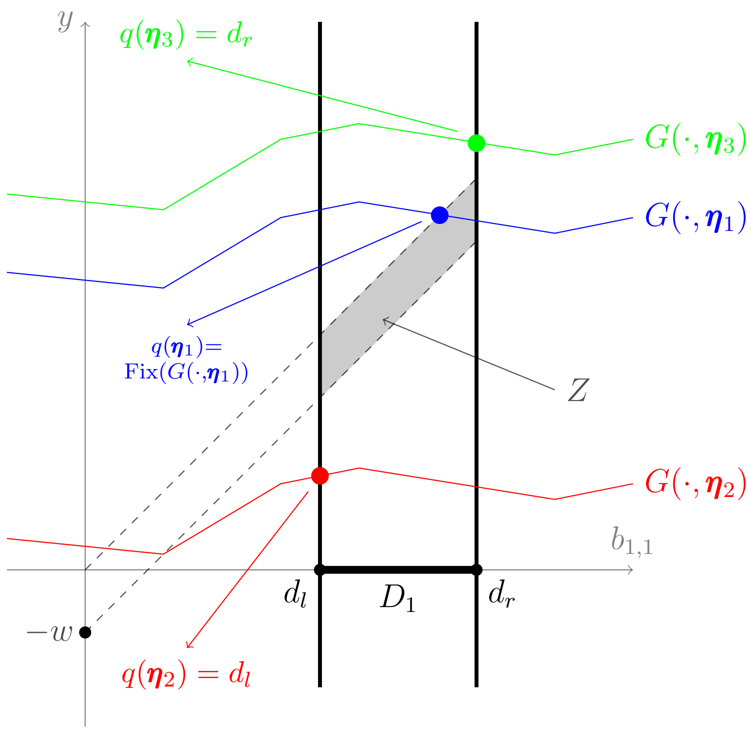

Figure 3.1: Visualization of the function $q(\boldsymbol{\eta})$ defined in (3.53). The continuous piecewise linear map $G(\cdot, \boldsymbol{\eta})$ is drawn for three different $\boldsymbol{\eta}$ values to show examples for all cases.

Claim 3.2.13. *The function $q : D_2 \to [d_l, d_r]$ has the following properties*

(a) *$q(\boldsymbol{\eta})$ is piecewise affine.*

(b) *There is a constant K_0 such that all of the partial derivatives of the function $q(\boldsymbol{\eta})$ are less than K_0 in absolute value.*

Part (a) is immediate from the formula (3.47). We obtain part (b) from (3.48) and (3.49) either by a direct calculation or by applying the Implicit Function Theorem. Now we put together (3.51), (3.52) and Claim 3.2.12 to obtain

$$K^{\boldsymbol{\eta}}_{k_1\dots k_n} \subset \left[q(\boldsymbol{\eta}), q(\boldsymbol{\eta}) + \frac{2(V-1)\rho_{k_1\dots k_n}}{1-\gamma} \right], \qquad \text{for all} \qquad \boldsymbol{\eta} \in D_2.$$

In this way $K_{k_1\dots k_n} \subset W_{k_1\dots k_n}$, where

$$W_{k_1\dots k_n} := \left\{ (b_{1,1}, \boldsymbol{\eta}) : \boldsymbol{\eta} \in D_2, b_{1,1} \in \left[q(\boldsymbol{\eta}), q(\boldsymbol{\eta}) + \frac{2(V-1)\rho_{k_1\dots k_n}}{1-\gamma} \right] \right\}. \tag{3.54}$$

Recall that $D = D_1 \times D_2$ is a cube and $D_1 = (d_l, d_r)$. That is D_2 is a translate of the cube $(0, d_r - d_l)^{L+m-1}$. Hence we can cover D_2 by $\left(\left\lceil \frac{d_r - d_l}{\rho_{k_1\dots k_n}} \right\rceil + 1 \right)^{L+m-1}$ pieces of $(L+m-1)$-dimensional r-mesh cubes, since $(k_1, \dots, k_n) \in \mathcal{M}_r$, and in this way

$$\rho_{\min} r < \rho_{k_1\dots k_n} \leq r. \tag{3.55}$$

Let Q be one of these $(L+m-1)$-dimensional r-mesh cubes. To count the number of $(L+m)$-dimensional r-mesh cubes needed to cover $\mathrm{Proj}_2^{-1}(Q) \cap W_{k_1\dots k_n}$, it is enough to calculate the lenght of its projection to D_1, as Q was already an r-mesh cube in one less dimensions. Pick an arbitrary $\boldsymbol{\eta} \in Q$, it defines a 1-dimensional slice in $\mathrm{Proj}_2^{-1}(Q) \cap W_{k_1\dots k_n}$. By (3.54), the length of this slice can never exceed $\frac{2(V-1)r}{1-\gamma}$, independently of the choice of $\boldsymbol{\eta}$. Further, the endpoints of such a slice depend on $q(\boldsymbol{\eta})$. Then it follows from part (b) of Claim 3.2.13 that the set $\mathrm{Proj}_2^{-1}(Q) \cap W_{k_1\dots k_n}$ can be covered by $(L+m-1) \cdot K_0 \cdot \frac{2(V-1)}{1-\gamma}$ many $(L+m)$-dimensional r-mesh cubes. That is we can cover $W_{k_1\dots k_n}$ (and consequently $K_{k_1\dots k_n}$) by

$$\left(\left\lceil \frac{d_r - d_l}{\rho_{k_1\dots k_n}} \right\rceil + 1 \right)^{L+m-1} \cdot (L+m-1) \cdot K_0 \cdot \frac{2(V-1)}{1-\gamma} \leq \frac{C_1}{r^{L+m-1}}$$

r-mesh cubes with a suitable constant C_1 independent of r and $(k_1, \dots, k_n) \in \mathcal{M}_r$, where in the last step we used (3.55). This completes the proof of the Lemma. □

As we already mentioned, Lemma 3.2.11 implies that (3.46) holds. That is, using (3.44) and (3.45), we can cover $E_{U,V} \cap D$ with at most $\frac{C_2}{r^{L+m-l+s_*}}$ $(L+m)$-dimensional r-mesh cubes. As it holds for all $D \subset \Delta$ and Δ is the countable union of such cubes, we just obtained (3.41). Since U and V were arbitrary, the proof of the Main Proposition follows.

□

3.3 Application to graph-directed IFSs

Here we are going to make explicit an important consequence of Jordan and Rapaport's result [14, Theorem 1.1] related to the dimension of the attractors of graph-directed self-similar systems on $\mathbb{R}$.

Given a self-similar graph-directed IFS $\mathcal{F}$ with a strongly connected graph. We will construct an ergodic and invariant measure μ on the symbolic space which is a Markov measure with the following property: the entropy of μ divided by the Lypunov exponent of μ is equal to $\alpha(\mathcal{F})$, where $\alpha(\mathcal{F})$ was defined in Definition 2.3.8. That is why μ can be considered as the natural measure for the self-similar graph-directed IFS $\mathcal{F}$.

Using this and the Jordan-Rapaport Theorem (Theorem 2.3.7), we obtain in Corollary 3.3.2, that the Hausdorff dimension of the push-forward measure of μ is the minimum of 1 and $\alpha(\mathcal{F})$ if ESC holds for the self-similar IFS $\mathcal{S}$ associated to $\mathcal{F}$. This can be considered as a generalization of part (b) of Hochman Theorem (Theorem 2.3.3) for graph-directed self-similar systems.

Throughout this section, we use the notation of Section 2.3.2. Consider a self-similar graph-directed iterated function system $\mathcal{F} = \{F_e(x) = r_e x + t_e\}_{e\in\mathcal{E}}$ with directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. We may identify

$$\mathcal{V} = [q] = \{1, \dots, q\}\,, \text{ and } \mathcal{E} = \{e_1, \dots, e_M\}\,.$$

We assume that $\mathcal{G}$ is strongly connected. Let $\mathcal{E}_\infty$ be the set of infinite length directed paths in $\mathcal{G}$

$$\mathcal{E}_\infty := \{\mathbf{e} = (e_1, e_2, \dots) : t(e_k) = s(e_{k+1}), \forall k \in \mathbb{N}\}\,.$$

Moreover, for $l \in [q]$ we introduce

$$\mathcal{E}_\infty^\ell := \{\mathbf{e} = (e_1, e_2, \dots) \in \mathcal{E}_\infty : s(e_1) = \ell\}\,.$$

We may identify $\mathcal{E}_\infty$ with $\Sigma_Z := \left\{(i_1, i_2, \dots) \in [M]^{\mathbb{N}} : z_{i_k, i_{k+1}} = 1,\ \forall k \in \mathbb{N}\right\}$, where $Z = (z_{i,j})_{i,j=1}^m$ is an $M \times M$ matrix such that

$$z_{i,j} = \begin{cases} 1, & \text{if } t(e_i) = s(e_j); \\ 0, & \text{otherwise.} \end{cases} \tag{3.56}$$

Set $\Pi_{\mathcal{F}}(e_1, e_2, \dots) := \lim_{n\to\infty} F_{e_1 \dots e_n}(0)$. It is clear that the non-empty compact sets $\left\{\Pi_{\mathcal{F}}(\mathcal{E}_\infty^\ell)\right\}_{\ell\in[q]}$ satisfy (2.10). Hence

$$\Lambda_i = \Pi_{\mathcal{F}}(\mathcal{E}_\infty^i), \text{ and } \Lambda = \Pi_{\mathcal{F}}(\mathcal{E}_\infty) \tag{3.57}$$

We define the natural pressure function as

$$\Phi(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{e}\in\mathcal{E}^n} |F_{\mathbf{e}}(\Lambda_{t(e_n)})|^s,$$

where the existence of the limit follows from the same standard argument used in the proof of Claim 3.2.8. We assumed that $\mathcal{G}$ is strongly connected, which implies that the matrix $\mathbf{C}^{(s)}$ we defined in (2.11) is irreducible. It is easy to see that by the Perron Frobenius Theorem we have

$$\Phi(s) = \log \varrho(\mathbf{C}^{(s)}),$$

where ϱ is the spectral radius. According to [16, Theorem 2] $\Phi(s)$ is continuous, strictly decreasing, $\Phi(0) > 0$, and $\lim_{s\to\infty} \Phi(s) = -\infty$, thus there exists a unique $0 < s_{\mathcal{F}}$ for which $\Phi(s_{\mathcal{F}}) = 0$. For a self-similar GDIFS with strongly connected graph we call $s_{\mathcal{F}}$ the natural dimension of the system. By Definition 2.3.8 it is clear that

$$s_{\mathcal{F}} = \alpha. \tag{3.58}$$

Definition 3.3.1. *We call* $\mathcal{S} = \{S_k(x) = r_{e_k}x + t_{e_k}\}_{k=1}^{M}$ ***the self-similar IFS associated with the self-similar GDIFS*** $\mathcal{F}$*. Clearly,*

$$S_k|_{\Lambda_{t(e_k)}} \equiv F_{e_k}|_{\Lambda_{t(e_k)}}.$$

Now we show that Part (a) of Theorem 2.3.9 combined with Jordan Rapaport Theorem (Theorem 2.3.7) implies that under some conditions, part (b) of Hochman's theorem holds for self-similar graph directed attractors as well, with $s_{\mathcal{F}}$ in the place of $\dim_{\mathrm{S}} \Lambda$.

Corollary 3.3.2. *Let* $\mathcal{F} = \{F_e\}_{e\in\mathcal{E}}$ *be a self-similar GDIFS on* $\mathbb{R}$ *with directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and attractor* Λ*. Further, assume that* $\mathcal{G}$ *is strongly connected, and that the self-similar IFS* $\mathcal{S}$ *associated to* $\mathcal{F}$ *satisfies the ESC. Then*

$$\dim_{\mathrm{H}} \Lambda = \min\{1, s_{\mathcal{F}}\}.$$

Proof. Since $\mathcal{G}$ is strongly connected, we can apply Theorem 2.3.9 to obtain

$$\dim_{\mathrm{H}} \Lambda \le \min\{\alpha, 1\}, \text{ since } \Lambda \subset \mathbb{R}. \tag{3.59}$$

In order to prove the opposite inequality, first we define an ergodic invariant measure μ on $\mathcal{V}^{\mathbb{N}}$. To do so, we recall the definition of α from Definition 2.3.8 and using that we consider the irreducible matrix $A = (a_{i,j})_{i,j=1}^{q} := \mathbf{C}^{(\alpha)}$. Let $\mathbf{u} = (u_1, \dots, u_q)$ and $\mathbf{v} = (v_1, \dots v_q)$ be the left and right eigenvectors of A

respectively, corresponding to the leading eigenvalue 1, normalized in a way that

$$\sum_{k=1}^{q} u_k v_k = 1, \quad \sum_{k=1}^{q} v_k = 1, \quad u_k, v_k > 0.$$

We define the stochastic matrix $P = (p(i,j))_{i,j=1}^{q}$ and its stationary distribution $\mathbf{p} = (p_1, \dots, p_q)$, related to the matrix A. That is

$$p(i,j) := \frac{a(i,j)v_j}{v_i}, \quad p_i := u_i \cdot v_i, \quad i,j \in [q].$$

Clearly, $\mathbf{p}$ is a probability vector and we have

$$\mathbf{p}^T \cdot P = \mathbf{p}^T.$$

With the help of these we can define the $(\mathbf{p}, P)$ Markov measure μ on n-cylinders $[v_1, \dots, v_n] \in \mathcal{V}^n$ as follows [27, p. 22]

$$\mu([v_1, \dots, v_n]) := p_{v_1} \cdot p_{v_1,v_2} p_{v_2,v_3} \cdots p_{v_{n-1},v_n}.$$

According to [27, Theorem 1.19] this extends to an ergodic, invariant measure on $\mathcal{V}^{\mathbb{N}}$. There is a natural bijection between $\mathrm{spt}(\mu) \subset \mathcal{V}^{\mathbb{N}}$ and $\mathcal{E}_\infty$, where $\mathrm{spt}(\cdot)$ denotes the support of the measure and

$$\mathrm{spt}(\mu) = \left\{(v_1, v_2, \dots) \in \mathcal{V}^{\mathbb{N}} : (v_i, v_{i+1}) \in \mathcal{E}, \text{ for all } i \in \mathbb{N}\right\}.$$

For a $(v_1, v_2, \dots) \in \mathrm{spt}(\mu)$ we define

$$\vartheta(v_1, v_2, \dots) := ((v_1, v_2), (v_2, v_3), \dots) \in \mathcal{E}_\infty,$$

where the inclusion holds since μ is only supported on vertex series $(v_1, v_2, \dots) \in \mathcal{V}^{\mathbb{N}}$ for which $(v_i, v_{i+1}) \in \mathcal{E}$ for all $i \geq 1$. Thus the pushforward measure $\nu := \vartheta_* \mu$ onto $\mathcal{E}_\infty$ is also ergodic and invariant with respect to the left shift σ. Observe that there is a natural embedding $\Psi : (\mathcal{E}_\infty, \sigma) \to \left([M]^{\mathbb{N}}, \sigma\right)$ defined as $\Psi(e_{i_1}, e_{i_2}, \dots) := (i_1, i_2, \dots)$, where we write σ also for left shift on $[M]^{\mathbb{N}}$. With the help of the matrix (3.56) we already defined $\Sigma_Z = \Psi(\mathcal{E}_\infty)$. The push forward measure $\Psi_* \nu$ is an invariant ergodic measure supported on Σ_Z.

Let $\Pi_{\mathcal{S}} : [M]^{\mathbb{N}} \to \mathbb{R}$ be the natural projection corresponding to $\mathcal{S}$. All the symbolic spaces and their relations we introduced so far are summarized on these diagrams:

$$\begin{array}{ccc} \mathrm{spt}(\mu) & \xrightarrow{\sigma} & \mathrm{spt}(\mu) \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\vartheta} \\ \mathcal{E}_\infty & \xrightarrow[\sigma]{} & \mathcal{E}_\infty \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ \Sigma_Z & \xrightarrow[\sigma]{} & \Sigma_Z \end{array} \qquad \begin{array}{ccc} \mathcal{E}_\infty & \xrightarrow{\Psi} & \Sigma_Z \\ & {\scriptstyle\Pi_{\mathcal{F}}}\searrow & \downarrow{\scriptstyle\Pi_{\mathcal{S}}} \\ & & \Lambda \end{array}$$

It is immediate that both diagrams are commutative. This and (3.57) imply that

$$\operatorname{spt}\left(\Pi_{\mathcal{S}*}(\Psi_*\nu)\right) = \Lambda. \tag{3.60}$$

Analogously to (3.23) and (3.24), easy calculations give us the entropy and the Lyapunov exponent of $\Psi_*\nu$:

$$\begin{aligned} h(\Psi_*\nu) &= -\lim_{n\to\infty} \frac{1}{n} \log \nu(\mathbf{e}|_n) = -\alpha \lim_{n\to\infty} \frac{1}{n} \log r_{\mathbf{e}|n} \\ \chi_{\Psi_*\nu} &= -\lim_{n\to\infty} \frac{1}{n} \log r_{\mathbf{e}|n}, \end{aligned}$$

for ν-almost all $\mathbf{e} = (e_1, e_2, \dots) \in \mathcal{E}_\infty$, where $r_{\mathbf{e}|n} = r_{e_1 \dots e_n}$ stands for the contraction ratio of $F_{\mathbf{e}|n}$. Hence we have

$$\frac{h(\Psi_*\nu)}{\chi_{\Psi_*\nu}} = \alpha. \tag{3.61}$$

We assumed that the ESC holds for $\mathcal{S}$, and we have seen that $\Psi_*\nu$ is an invariant and ergodic probability measure. Thus we can apply the Jordan Rapaport Theorem (Theorem 2.3.7) to obtain that

$$\dim_{\mathrm{H}} \Lambda \geq \min\{\alpha, 1\}, \tag{3.62}$$

by substituting (3.60) and (3.61) into the theorem.

Together (3.62) and (3.59) yields

$$\min\{\alpha, 1\} \leq \dim_{\mathrm{H}} \Lambda \leq \min\{\alpha, 1\}. \tag{3.63}$$

According to (3.58) $s_{\mathcal{F}} = \alpha$, hence (3.63) implies

$$\dim_{\mathrm{H}} \Lambda = \min\{s_{\mathcal{F}}, 1\}.$$

□

Chapter 4

Equality of dimensions

In this chapter, we are going to prove the following, improved version of the first half of Theorem 3.2.1.

Theorem 4.0.1. *Let $\mathcal{F}$ be a* $\dim_{\mathrm{P}}$*-typical CPLIFS, then*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\{1, s_{\mathcal{F}}\}.$$

Unlike in the case of small CPLIFSs, it is an open question wether the Hausdorff dimension of the attractor Λ is equal to the Hausdorff dimension of an ergodic invariant measure of the IFS.

Using Markov diagrams, the proof of this theorem will also show how to obtain invariant measures whose Hausdorff dimension approximates $\dim_{\mathrm{H}} \Lambda$.

4.1 Markov Diagrams

P. Raith and F. Hofbauer proved results on the dimension of expanding piecewise monotonic systems using the notion of Markov diagrams [22, 24, 11]. We will define the Markov diagram in a similar fashion for CPLIFSs, and then use it to prove that $s_{\mathcal{F}}$ equals the Hausdorff dimension of the attractor for non-regular systems as well, under some weak assumptions [20].

Building Markov diagrams

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS, and let I be the interval defined by (1.2). We write $I_k := f_k(I)$ and $\mathcal{I} = \cup_{k=1}^m I_k$. For $k \in [m], j \in [l(k)+1]$, we define $f_{k,j} : J_{k,j} \to I_k$ as the unique linear function that satisfies $\forall x \in J_{k,j} : f_k(x) = f_{k,j}(x)$. We call the expansive linear functions

$$\begin{aligned} \forall k \in [m], \forall j \in & [l(k)+1] : \quad f_{k,j}^{-1} : f_k(J_{k,j}) \to J_{k,j}, \\ & \forall x \in J_{k,j} : f_{k,j}^{-1}(f_k(x)) = x \end{aligned}$$

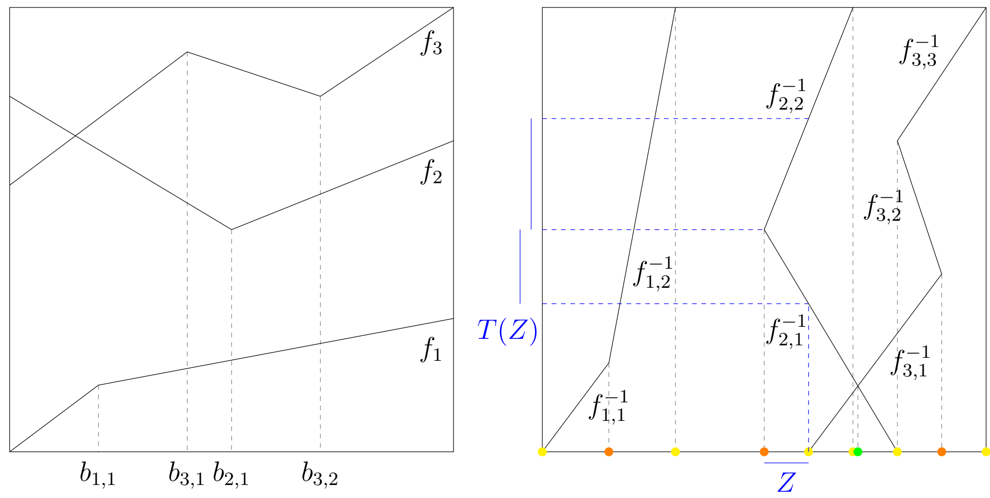

Figure 4.1: A CPLIFS $\mathcal{F} = \{f_k\}_{k=1}^m$ is on the left with its associated expansive multi-valued mapping T on the right. The critical points are marked according to the color scheme of Definition 4.1.1

the **branches** of the multi-valued mapping T. As the notation suggests, these are the local inverses of the elements of $\mathcal{F}$.

We define the **expanding multi-valued mapping associated to $\mathcal{F}$** as

$$T : \mathcal{I} \mapsto \mathcal{P}(\mathcal{P}(I)), \quad T(x) := \left\{\{f_{k,j}^{-1}(x)\}\right\}_{k\in[m],j\in[l(k)]}.$$

That is the image of any Borel subset $A \subset \mathcal{I}$ is

$$T(A) = \left\{\{f_{k,j}^{-1}(x) : x \in A\}\right\}_{k\in[m],j\in[l(k)]}.$$

Note that instead of forming an union, we differentiate between the preimages produced by different branches. This will enable us to investigate the consequences of having overlappings in the construction.

Definition 4.1.1. *We define the* ***set of critical points*** *as*

$$\mathcal{K} := \cup_{k=1}^m \{f_k(0), f_k(1)\} \bigcup \cup_{k=1}^m \cup_{j=1}^{l(k)} f_k(b_{k,j}) \bigcup$$
$$\left\{x \in \mathcal{I} \middle| \exists k_1, k_2 \in [m], \exists j_1 \in [l(k_1)], \exists j_2 \in [l(k_2)], k_1 \neq k_2 : f_{k_1,j_1}^{-1}(x) = f_{k_2,j_2}^{-1}(x)\right\}.$$

Definition 4.1.2. *We call the partition of $\mathcal{I}$ into closed intervals defined by the set of critical points $\mathcal{K}$ the* ***monotonicity partition*** *$\mathcal{Z}_0$ of $\mathcal{F}$. We call its elements* ***monotonicity intervals****.*

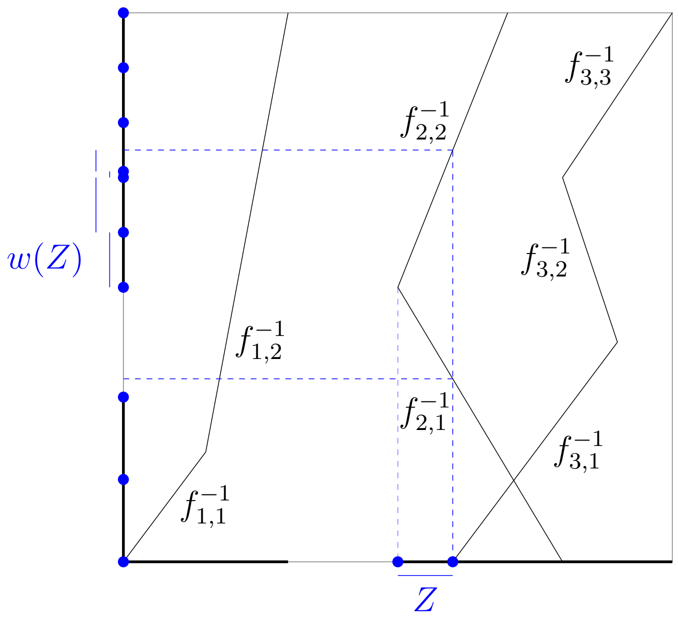

Figure 4.2: This figure shows how the set of successors $w(Z)$ is obtained for an interval Z. The blue points on the y-axis are denoting the monotonicity partition.

Definition 4.1.3. *Let $Z \in \mathcal{Z}_0$. We say that the interval D is a* ***successor*** *of Z and we write $Z \to D$ if*

$$\exists Z_0 \in \mathcal{Z}_0, Z' \in T(Z) : D = Z_0 \cap Z'.$$

Further, we write $Z \to_{k,j} D$ if

$$\exists Z_0 \in \mathcal{Z}_0 : D = Z_0 \cap f_{k,j}^{-1}(Z).$$

The set of successors of Z is denoted by $w(Z) := \{D | Z \to D\}$.

Similarly, we define $w(\mathcal{Z}_0)$ as the set of the successors of all elements of $\mathcal{Z}_0$. That is

$$w(\mathcal{Z}_0) := \cup_{Z \in \mathcal{Z}_0} w(Z).$$

Definition 4.1.4. *Let $\mathcal{Z}_0'$ be a finite refinement of the partition $\mathcal{Z}_0$. We say that $(\mathcal{D}, \to)$ is the* ***Markov Diagram of $\mathcal{F}$ with respect to $\mathcal{Z}_0'$*** *if $\mathcal{D}$ is the smallest set containing $\mathcal{Z}_0'$ that satisfies*

$$\forall C \in \mathcal{D} : C \to D \implies D \in \mathcal{D}.$$

When $\mathcal{Z}_0' = \mathcal{Z}_0$ we simply call $(\mathcal{D}, \to)$ the Markov diagram of $\mathcal{F}$.

We often use the notation

$$\mathcal{D}_n := \cup_{i=0}^{n} w^i(\mathcal{Z}_0), \text{ where } w^i(\mathcal{Z}_0) = \underbrace{w \circ \cdots \circ w}_{i \text{ times}}(\mathcal{Z}_0).$$

Obviously,

$$\mathcal{D} = \cup_{i\geq 0} w^i(\mathcal{Z}_0). \tag{4.1}$$

If the union in (4.1) is finite, we say that **the Markov diagram is finite**. We define recursively the ***n*-th level of the Markov diagram** as

$$\mathcal{Z}_n := w(\mathcal{Z}_{n-1}) \setminus \cup_{i=0}^{n-1} \mathcal{Z}_i,$$

for $n \geq 1$.

One can imagine the Markov diagram as a (potentially infinitely big) directed graph, with vertex set $\mathcal{D}$. Between $C, D \in \mathcal{D}$, we have a directed edge $C \to D$ if and only if $D \in w(C)$. We call the Markov diagram **irreducible** if there exists a directed path between any two intervals $C, D \in \mathcal{D}$. In the next lemma we prove that by choosing an appropriate refinement $\mathcal{Y}_0$ of $\mathcal{Z}_0$, the Markov diagram $(\mathcal{D}', \to)$ of $\mathcal{F}$ with respect to $\mathcal{Y}_0$ always has an irreducible subdiagram. Further, the elements of $\mathcal{D}'$ cover Λ. It implies that $(\mathcal{D}', \to)$ is sufficient to describe the orbits of the points of Λ. That is, Lemma 4.1.5 enables us to assume that the Markov diagram $(\mathcal{D}, \to)$ of $\mathcal{F}$ is irreducible without loss of generality.

Lemma 4.1.5. *Let $\mathcal{F}$ be a CPLIFS with attractor Λ, and let $(\mathcal{D}(\mathcal{Y}_0), \to)$ be its Markov diagram with respect to some finite refinement $\mathcal{Y}_0$ of the monotonicity partition $\mathcal{Z}_0$. For the right choice of $\mathcal{Y}_0$, there exists an irreducible subdiagram $(\mathcal{D}', \to)$ of $(\mathcal{D}(\mathcal{Y}_0), \to)$, such that the elements of $\mathcal{D}'$ cover Λ.*

Proof. For $k \in [m]$, let ϕ_k be the fixed point of f_k. We assume without loss of generality that $\phi_i \leq \phi_j$ if $i < j$ for $i, j \in [m]$. Let $\mathcal{Y}_0$ be the refinement of $\mathcal{Z}_0$ with ϕ_1. That is, $\mathcal{Y}_0$ is the partition of $\mathcal{I}$ defined by $\mathcal{K} \cup \{\phi_1\}$.

There are at most two intervals in $\mathcal{Y}_0$ that ends in ϕ_1, we write Y_1 and Y_2 for them. Let $\mathcal{D}'$ be the set that contains Y_1, Y_2 and all of their successors which intersect Λ. In particular,

$$\mathcal{D}' := \{Y_1, Y_2\} \bigcup \{C \subset \mathcal{I} | \exists n > 0, \exists j \in \{1, 2\} : C \in w^n(Y_i) \,\&\, C \cap \Lambda \neq \emptyset\}.$$

Obviously, $(\mathcal{D}', \to)$ is a subdiagram of $(\mathcal{D}(\mathcal{Y}_0), \to)$. Let $C \in \mathcal{D}'$ be an arbitrary interval. The attractor Λ is the invariant set of $\mathcal{F}$, thus some element of $w(C)$ also intersects Λ. It follows that there are no deadends in $(\mathcal{D}', \to)$, every directed path can be continued within the subdiagram. That is

$$\forall C \in \mathcal{D}', \exists D \in \mathcal{D}' : C \to D.$$

Since $C \cap \Lambda \neq \emptyset$ and the elements of $\mathcal{F}$ are strict contractions, there exist $N > 0$ and $\mathbf{k} = (k_1, \dots, k_N) \in \{1, \dots, m\}^N$ such that $f_{\mathbf{k}}(I) \subset C$. Then, the elements of the set

$$\{C_N \in \mathcal{D}' | \exists j_1, \dots, j_N : C \to_{k_1, j_1} \cdots \to_{k_N, j_N} C_N\}$$

cover I and hence the attractor Λ. We just obtained that

$$\forall C \in \mathcal{D}', \forall U \subset \mathcal{I} : \exists n > 0, \exists C' \in w^n(C) \text{ such that } C' \cap U \neq \emptyset. \tag{4.2}$$

By applying (4.2) to a U neighbourhood of ϕ_1, it follows that from every $C \in \mathcal{D}'$ there is a directed path in $(\mathcal{D}', \to)$ to some $C' \in \mathcal{D}'$ that ends in ϕ_1.

Since ϕ_1 is the fixed point of the expansive mapping f_1^{-1}, if $\phi_1 \in C \in \mathcal{D}'$, then there must be a directed path in $(\mathcal{D}', \to)$ from C to either Y_1 or Y_2. By applying (4.2) again, there are directed paths in $(\mathcal{D}', \to)$ between Y_1 and Y_2, thus it is indeed an irreducible subdiagram. $\square$

4.1.1 Connection to the natural pressure

Similarly to graph-directed iterated function systems, we associate a matrix to Markov diagrams, that will help us determine the Hausdorff dimension of the corresponding CPLIFS (see [8]).

Definition 4.1.6. *Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS, and write $(\mathcal{D}, \to)$ for its Markov diagram. We define the matrix $\mathbf{F}(s) := \mathbf{F}_{\mathcal{D}}(s)$ indexed by the elements of $\mathcal{D}$ as*

$$[\mathbf{F}(s)]_{C,D} := \begin{cases} \sum_{(k,j): C \to_{(k,j)} D} |f'_{k,j}|^s, & \text{if } C \to D \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

*We call $\mathbf{F}_{\mathcal{D}}(s)$ the **matrix associated to the Markov diagram** $(\mathcal{D}, \to)$.*

We used in the definition, that for a $D \in \mathcal{D}$ with $C \to_{(k,j)} D$ the derivative of $f_{k,j}$ over D is a constant number. That is each element of $\mathbf{F}(s)$ is either zero or a sum of the s-th power of some contraction ratios.

This matrix can be defined for any $\mathcal{C} \subset \mathcal{D}$ as well, by choosing the indices from $\mathcal{C}$ only. We write $\mathbf{F}_{\mathcal{C}}(s)$ for such a matrix. It follows that $\mathbf{F}_{\mathcal{C}}(s)$ is always a submatrix of $\mathbf{F}(s)$ for $\mathcal{C} \subset \mathcal{D}$.

We write $\mathcal{E}_{\mathcal{C}}(n)$ for the set of n-length directed paths in the graph $(\mathcal{C}, \to)$.

$$\begin{aligned} \mathcal{E}_{\mathcal{C}}(n) := &\{((k_1, j_1), \dots, (k_n, j_n)) \big| \exists C_1, \dots, C_{n+1} \in \mathcal{C} : \\ &\forall q \in [n], \exists k_q \in [m], j_q \in [l(k)] : C_q \to_{(k_q, j_q)} C_{q+1}\}. \end{aligned}$$

An n-length directed path here means n many consecutive directed edges, and we identify each such path with the labels of the included edges in order. Each path in $(\mathcal{D}, \rightarrow)$ of infinite length represents a point in Λ, and each point is represented by at least one path. Similarly, for $\mathcal{C} \subset \mathcal{D}$ the points defined by the natural projection of infinite paths in $(\mathcal{C}, \rightarrow)$ form an invariant set $\Lambda_{\mathcal{C}} \subset \Lambda$. We define the natural pressure of these sets as

$$\Phi_{\mathcal{C}}(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{k}} |I_{\mathbf{k}}|^s, \tag{4.4}$$

where the sum is taken over all $\mathbf{k} = (k_1, \ldots k_n)$ for which $\exists j_1, \ldots j_n : ((k_1, j_1), \ldots, (k_n, j_n)) \in \mathcal{E}_{\mathcal{C}}(n)$, and I is the interval defined in (1.1). By the definition of $\mathcal{D}$ it is easy to see that $\Phi_{\mathcal{D}}(s) = \Phi(s)$.

Remark 4.1.7. *Let $\mathcal{Y}_0$ be a finite refinement of the monotonicity partition $\mathcal{Z}_0$, and let $(\mathcal{D}', \rightarrow)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}_0$. Obviously,*

$$\forall s \geq 0 : \Phi_{\mathcal{D}}(s) = \Phi_{\mathcal{D}'}(s).$$

Namely, the summation on the right hand side of (4.4) *is taken for the same set of words when we compute $\Phi_{\mathcal{D}}(s)$ and $\Phi_{\mathcal{D}'}(s)$.*

We will show, that the unique zero of the function $\Phi_{\mathcal{D}}(s)$ can be approximated by the root of $\Phi_{\mathcal{C}}(s)$ for some $\mathcal{C} \subset \mathcal{D}$. To show this, we need to connect the function $\Phi_{\mathcal{C}}(s)$ to the matrix $\mathbf{F}_{\mathcal{C}}(s)$.

As an operator, $(\mathbf{F}_{\mathcal{D}}(s))^n$ is always bounded in the l^∞-norm. Thus we can define

$$\varrho(\mathbf{F}_{\mathcal{C}}(s)) := \lim_{n\to\infty} \|(\mathbf{F}_{\mathcal{C}}(s))^n\|_\infty^{1/n}.$$

Lemma 4.1.8. *Let $\mathcal{C} \subset \mathcal{D}$. If $(\mathcal{C}, \rightarrow)$ is irreducible, then*

$$\Phi_{\mathcal{C}}(s) \leq \log \varrho(\mathbf{F}_{\mathcal{C}}(s)). \tag{4.5}$$

If $(\mathcal{C}, \rightarrow)$ is irreducible and finite, then

$$\Phi_{\mathcal{C}}(s) = \log \varrho(\mathbf{F}_{\mathcal{C}}(s)). \tag{4.6}$$

Proof. First only assume that $(\mathcal{C}, \rightarrow)$ is irreducible. Since it is irreducible, we can think about $(\mathcal{C}, \rightarrow)$ as the Markov diagram of some IFS with level n cylinder intervals $\{I_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{E}_{\mathcal{C}}(n)}$.

Fix $\mathbf{k} = (k_1, \ldots, k_n) \in [m]^n$. There are at least one, but possibly several directed paths of length n in the graph with labels $((k_1, j_1), \ldots, (k_n, j_n))$ for some $j_1, \ldots j_n$. Each of these paths corresponds to a unique entry in $\mathbf{F}^n_{\mathcal{C}}(s)$.

The biggest one of these entries times $|I|$ is an upper bound on $|I_{\mathbf{k}}|^s$. Since every n length path starts at some element of $\mathcal{Z}_0$, we obtain that

$$\sum_{\mathbf{k}} |I_{\mathbf{k}}|^s \leq |\mathcal{Z}_0| \cdot \|\mathbf{F}^n_{\mathcal{C}}(s)\|_\infty \cdot |I|, \tag{4.7}$$

where the sum is taken over all $\mathbf{k} = (k_1, \dots k_n)$ for which $\exists j_1, \dots j_n : ((k_1, j_1), \dots, (k_n, j_n)) \in \mathcal{E}_{\mathcal{C}}(n)$. By taking logarithm on both sides, dividing them by n, and then taking the limit as $n \to \infty$, we obtain (4.5).

Now assume that $\mathcal{C}$ is finite, and write N for the highest level of the Markov diagram. It means that for every $|\mathbf{k}| \geq N$, the cylinder interval $I_{\mathbf{k}}$ is contained in an element of $\mathcal{C}$. That is

$$\forall n \geq 0 : \min_{|\mathbf{l}|=N} |I_{\mathbf{l}}| \|\mathbf{F}^n_{\mathcal{C}}(s)\|_\infty \leq \sum_{\mathbf{k}} |I_{\mathbf{k}}|^s \leq \max_{|\mathbf{l}|=N} |I_{\mathbf{l}}| |\mathcal{Z}_0| \|\mathbf{F}^n_{\mathcal{C}}(s)\|_\infty, \tag{4.8}$$

where the sum in the middle is taken over all $\mathbf{k} = (k_1, \dots k_{N+n})$ for which

$$\exists j_1, \dots j_{N+n} : ((k_1, j_1), \dots, (k_{N+n}, j_{N+n})) \in \mathcal{E}_{\mathcal{C}}(N+n).$$

It follows that (4.6) holds.

□

Let $(\mathcal{C}_1, \to), (\mathcal{C}_2, \to), \dots$ be an increasing sequence of irreducible subgraphs of $(\mathcal{D}, \to)$. It follows from Seneta's results [25, Theorem 1] that the so called R-values of the matrices $\mathbf{F}_{\mathcal{C}_n}(s)$ converge to the R-value of $\mathbf{F}(s)$. For an irreducible finite matrix $\mathbf{A}$ we always have $R(\mathbf{A}) = \frac{1}{\varrho(\mathbf{A})}$, then the convergence of the spectral radius $\varrho(\mathbf{F}_{\mathcal{C}_n}(s))$ to $\varrho(\mathbf{F}(s))$ follows. Altough $\mathbf{F}(s)$ may not be finite, the relation $R(\mathbf{F}(s)) = \frac{1}{\varrho(\mathbf{F}(s))}$ can still be guaranteed by some assumptions. That is why the following property has a crucial role in our proofs.

Definition 4.1.9. *Let $\mathcal{F}$ be a CPLIFS and $\mathcal{Y}$ be a finite refinement of the monotonicity partition $\mathcal{Z}_0$. Let $(\mathcal{D}(\mathcal{Y}), \to)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$, and let $\mathbf{F}(\mathcal{Y}, s)$ be its associated matrix.*

*We say that the CPLIFS $\mathcal{F}$ is **limit-irreducible** if there exists a $\mathcal{Y}$ such that for all $s \in (0, \dim_{\mathrm{H}} \Lambda]$ the matrix $\mathbf{F}(\mathcal{Y}, s)$ has right and left eigenvectors with nonnegative entries for the eigenvalue $\varrho(\mathbf{F}(\mathcal{Y}, s))$.*

*We call this finite partition $\mathcal{Y}$ a **limit-irreducible partition of $\mathcal{F}$** and $(\mathcal{D}(\mathcal{Y}), \to)$ a **limit-irreducible Markov diagram of $\mathcal{F}$**.*

In the next section we show how being limit-irreducible implies that the Hausdorff dimension of the attractor is equal to the minimum of the natural dimension and 1. Later, in Section 4.3, we investigate what makes a CPLIFS limit-irreducible.

4.2 Proof using the diagrams

We have already shown a connection between the Markov diagram and the natural pressure of a given CPLIFS. Now using this connection, we show that the natural dimension of a limit-irreducible CPLIFS is always a lower bound for the Hausdorff dimension of its attractor, by approximating the spectral radius of the Markov diagram with its submatrices' spectral radius.

As in [22], the following proposition holds.

Proposition 4.2.1. *Let $\mathcal{F}$ be a limit-irreducible CPLIFS, and let $(\mathcal{D}, \rightarrow)$ be its limit-irreducible Markov diagram. For any $\varepsilon > 0$ there exists a $\mathcal{C} \subset \mathcal{D}$ finite subset such that*

$$\varrho(\mathbf{F}(s)) - \varepsilon \leq \varrho(\mathbf{F}_{\mathcal{C}}(s)) \leq \varrho(\mathbf{F}(s)), \tag{4.9}$$

where $\mathbf{F}(s)$ is the matrix associated to $(\mathcal{D}, \rightarrow)$.

The proof is essentially the same as the proof of [22, Lemma 6 (ii)]. We obtain the following theorem as the combinations of [22, Theorem 2] and Theorem 3.3.2.

Theorem 4.2.2. *Let $\mathcal{F}$ be a limit-irreducible CPLIFS with attractor Λ and limit-irreducible Markov diagram $(\mathcal{D}, \rightarrow)$. Assume that the generated self-similar system of $\mathcal{F}$ satisfies the ESC. Then*

$$\dim_H \Lambda = \min\{1, s_{\mathcal{F}}\},$$

where $s_{\mathcal{F}}$ denotes the unique zero of the natural pressure function $\Phi(s)$.

The proof is similar to the proof of Theorem 2 in [22].

Proof. By Corollary 1.0.1, $\dim_H \Lambda \leq \min\{s_{\mathcal{F}}, 1\}$ always holds. It is only left to prove the lower bound.

Choose an arbitrary $t \in (0, s_{\mathcal{F}})$. The natural pressure function is strictly decreasing and has a unique zero at $s_{\mathcal{F}}$, hence $\Phi(t) > 0$. The same can be told about the spectral radius of $\mathcal{D}$, according to Remark 4.1.7. $(\mathcal{D}, \rightarrow)$ is irreducible, but not necessarily finite, thus Lemma 4.1.8 gives

$$0 < \Phi(t) \leq \log \varrho\left(\mathbf{F}(t)\right).$$

Pick an arbitrary $\varepsilon \in (0, \varrho\left(\mathbf{F}(t)\right) - 1)$. According to Proposition 4.2.1, there exists a $\mathcal{C} \subset \mathcal{D}$ subset satisfying (4.9) for this ε. That is

$$\exists\, \mathcal{C} \subset \mathcal{D} \text{ finite}: \ \log \varrho(\mathbf{F}_{\mathcal{C}}(t)) > 0.$$

Then applying Lemma 4.1.8 again gives

$$0 < \log \varrho(\mathbf{F}_{\mathcal{C}}(t)) = \Phi_{\mathcal{C}}(t), \tag{4.10}$$

since $\mathcal{C}$ is finite.

For a finite $\mathcal{C}$ the induced attractor $\Lambda_{\mathcal{C}}$ belongs to a graph-directed IFS with directed graph $(\mathcal{C}, \to)$. Since the generated self-similar system satisfies the ESC, we already know from Corollary 3.3.2 that

$$\dim_H \Lambda_{\mathcal{C}} = \min\{s_{\mathcal{C}}, 1\}, \tag{4.11}$$

where $s_{\mathcal{C}}$ is the unique root of $\Phi_{\mathcal{C}}(s)$.

Assume first that $s_{\mathcal{F}} \leq 1$, which implies $s_{\mathcal{C}} \leq 1$ for all $\mathcal{C} \subset \mathcal{D}$. Together (4.11) and (4.10) yields

$$0 < \Phi_{\mathcal{C}}(t) \implies t < s_{\mathcal{C}} = \dim_H \Lambda_{\mathcal{C}} \leq \dim_H \Lambda,$$

and it holds for any $t \in (0, s_{\mathcal{F}})$. Thus $s_{\mathcal{F}} \leq \dim_{\mathrm{H}} \Lambda$.

When $s_{\mathcal{F}} > 1$, we can find a $\mathcal{C} \subset \mathcal{D}$ for which $\dim_H \Lambda_{\mathcal{C}} = 1$. It is a simple consequence of Lemma 4.1.8, Proposition 4.2.1 and (4.11). Therefore the lower bound that covers both cases is

$$\min\{s_{\mathcal{F}}, 1\} \leq \dim_{\mathrm{H}} \Lambda.$$

□

Remark 4.2.3. *As for all finite subdiagram $(\mathcal{C}, \to)$ the set $\Lambda_{\mathcal{C}}$ is a subset of Λ, and by Corollary 3.3.2 we can always find an invariant ergodic measure supported on $\Lambda_{\mathcal{C}}$ satisfying $\dim_{\mathrm{H}} \Lambda_{\mathcal{C}} = \dim_{\mathrm{H}} \mu$, we also obtained that*

$$\dim_{\mathrm{H}} \Lambda = \sup\{\dim_{\mathrm{H}} \mu : \mu \textit{ is ergodic and invariant}, \operatorname{supp}(\mu) \subset \Lambda\},$$

if $\mathcal{F}$ is limit-irreducible.

The proof of Theorem 4.2.2 also implies the following.

Corollary 4.2.4. *Let $\mathcal{F}$ be a limit-irreducible CPLIFS with limit-irreducible Markov diagram $(\mathcal{D}, \to)$ and attractor Λ. Let $\mathbf{F}(\mathbf{s})$ be the matrix associated to $(\mathcal{D}, \to)$. Then for all $s \in (0, s_{\mathcal{F}}]$*

$$\log \varrho(\mathbf{F}(s)) = \Phi(s).$$

We are left to prove that limit-irreducibility is a $\dim_{\mathrm{P}}$-typical property of CPLIFSs.

4.3 What makes a CPLIFS limit-irreducible?

It is hard to check if a CPLIFS $\mathcal{F} = \{f_k\}_{k=1}^m$ is limit-irreducible, that is if it satisfies definition 4.1.9. In this section, we verify the following proposition by a case analysis.

Proposition 4.3.1. *Let $\mathcal{F}$ be a CPLIFS with generated self-similar system $\mathcal{S}$. If $\mathcal{S}$ satisfies the ESC, then $\mathcal{F}$ is limit-irreducible.*

Theorem 4.0.1 is a straightforward consequence of Proposition 4.3.1 and Theorem 4.2.2.

According to [11, Corollary 1], if all functions in $\mathcal{F}$ are injective and the first cylinders are not overlapping, then $\mathcal{F}$ is limit-irreducible. This observation was utilized by Raith in the proof of [22, Lemma 6].

In this section we always assume that $s \in (0, \dim_{\mathrm{H}} \Lambda]$. The overlapping structures may induce multiple edges in the Markov diagram. In the associated matrix $\mathbf{F}(s)$ each multiple edge is represented as an entry of the form $\rho_{k_1,j_1}^s + \cdots + \rho_{k_n,j_n}^s$ for some $n > 1$. Since these entries can be bigger than 1 in absolute value, the assumptions of [11, Corollary 1] do not hold. We need to investigate under which conditions can [11, Corollary 1/ii] help us.

Lemma 4.3.2 (F. Hofbauer [11, Corollary 1/ii])**.** *Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS with monotonicity partition $\mathcal{Z}_0$ and $\mathcal{Y}$ be a finite refinement of $\mathcal{Z}_0$. Let $(\mathcal{D}, \to)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$ and $\mathbf{F}(s)$ be its associated matrix. If $\mathbf{F}(s)$ can be written in the form*

$$\mathbf{F}(s) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

such that $\varrho(\mathbf{F}(s)) > \varrho(S)$, then $\mathcal{F}$ is limit-irreducible with limit-irreducible partition $\mathcal{Y}$. Here P, Q, R, S are appropriate dimensional block matrices.

For the convenience of the reader we also present Hofbauer's proof here.

Proof. We follow the proof of Corollary 1/ii right after Theorem 9 in [11]. Let $\lambda := \varrho(\mathbf{F}(s))$ and I_d be the d dimensional identity matrix. We write d_P and d_F for the dimensions of the square matrices P and $\mathbf{F}(s)$ respectively. It follows that $d_S = d_F - d_P$. We remark that d_F and d_S may not be finite.

As $\lambda > \varrho(S)$, $(I_{d_S} - xS)^{-1} = \sum_{k=0}^{\infty} x^k S^k$ exists for $|x| \le \lambda^{-1}$ and has nonnegative entries for $0 \le x \le \lambda^{-1}$. For $E(x) = P + xQ(I - xS)^{-1}R$ we have the following matrix equation

$$\begin{bmatrix} I - xE(x) & -xQ(I - xS)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -xR & I - xS \end{bmatrix} = I - x\mathbf{F}(s), \qquad (4.12)$$

for all $|x| \le \lambda^{-1}$. Since $\lambda = \varrho(\mathbf{F}(s))$, we find an x with $|x| = \lambda^{-1}$ such that $I - x\mathbf{F}(s)$ is not invertible. Fix this x number. By (4.12), knowing that both I and $(I - xS)^{-1}$ are invertible, we get that $I - xE(x)$ is not invertible, i.e. $\varrho(E(x)) \ge \lambda$. Since the entries of $E(|x|)$ are greater than or equal to the absolute values of the entries of $E(x)$, we get $\varrho(E(\lambda^{-1})) = \varrho(E(|x|)) \ge \varrho(E(x)) \ge \lambda$. Note that $E(x)$ is a finite matrix.

For $t \in (0, \lambda^{-1}]$ the map $t \to \varrho(E(t))$ is continuous and increasing, since the entries of $E(t)$ are continuous and increasing in t. Since $\varrho(E(\lambda^{-1})) \ge \lambda$, we find a $y \in (0, \lambda^{-1}]$ with $\varrho(E(y)) = y^{-1}$. Since $E(y)$ has nonnegative entries, this implies that $I - yE(y)$ is not invertible. Hence $I - y\mathbf{F}(s)$ is not invertible by (4.12). As $\lambda = \varrho(\mathbf{F}(s))$, we get $y = \lambda^{-1}$. Since $E(y)$ is a finite matrix, we find a nonnegative vector u_1 with $u_1(I - yE(y)) = 0$. Set $u_2 = yu_1Q(I - yS)^{-1}$, which is a nonnegative $l^1(d_S)$ vector, as the rows of Q are in $l^1(d_S)$. Hence $u = (u_1, u_2)$ is a nonnegative $l^1(d_F)$ vector and $u(I - y\mathbf{F}(s)) = 0$ by (4.12). That is u is a left eigenvector for $\lambda = \varrho(\mathbf{F}(s))$.

Similar calculation for the transpose of $\mathbf{F}(s)$ yield a nonnegative $l^\infty(d_F)$ vector v with $(I - \lambda^{-1}\mathbf{F}(s))v = 0$.

□

Lemma 4.1.8 implies that for $s \in (0, \dim_{\mathrm{H}} \Lambda)$ we have $\varrho(\mathbf{F}(s)) > 1$, where Λ is the attractor of the CPLIFS $\mathcal{F}$. Therefore, in order to apply Lemma 4.3.2, it is enough to show that

$$\lim_{N\to\infty} \varrho\left(\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)\right) = 1. \tag{4.13}$$

If (4.13) holds, then $\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)$ can take the place of the submatrix S in Theorem 4.3.2 for a big enough N.

In the special case of expansive piecewise monotonic mappings, (4.13) was verified by F. Hofbauer [11, Corollary 1/i]. To extend his results to CPLIFS, we need to show the same for our expansive multi-valued mappings T. The only difference between our and his Markov diagrams is the occurence of multiple edges, caused by the possible overlappings. We note that not all of the overlappings induce multiple edges, as monotonicity intervals of the same level might overlap.

Definition 4.3.3. *Let $Z \in \mathcal{Z}$ be an element of the base partition, and let $f^{-1}_{k_1,j_1}, f^{-1}_{k_2,j_2}$ be two different branches of the expansive multivalued mapping T. We say that $f^{-1}_{k_1,j_1}$ and $f^{-1}_{k_2,j_2}$* ***cause an overlap on*** *Z if*

$$int(f^{-1}_{k_1,j_1}(Z)) \cap int(f^{-1}_{k_2,j_2}(Z)) \neq \emptyset,$$

where $int(A)$ denotes the inerior of the set A. If $\exists z \in Z : f^{-1}_{k_1,j_1}(z) = f^{-1}_{k_2,j_2}(z)$, then we call it a ***cross overlap****, otherwise we call it a* ***light overlap****. See*

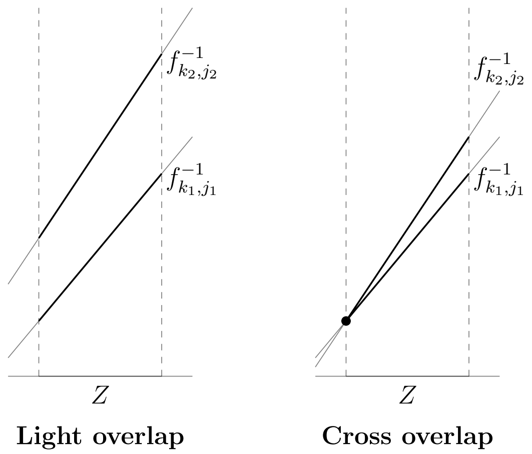

Figure 4.3: The two types of overlappings.

Figure 4.3. We call the branches that cause an overlap over the same interval ***cross overlapping branches*** *or* ***light overlapping branches****, respectively.*

Note that the graphs of the branches of T can only intersect at the endpoint of some base interval $Z \in \mathcal{Z}_0$ (see Definition 4.1.2). We say that the **order of overlapping** is K if the maximal number of branches of T that have intersecting domains is K.

4.3.1 The case of light overlaps

Lemma 4.3.4. *Let $\mathcal{F}$ be a CPLIFS with only light overlaps. Then, there exists a finite partition $\mathcal{Y}$ such that the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$ do not contain any multiple edges.*

Proof. Let K be the order of overlapping of $\mathcal{F}$ and T be the multi-valued mapping associated to $\mathcal{F}$. First assume that the branches of T overlap only above $Z \in \mathcal{Z}_0$ and write $f^{-1}_{k_1,j_1}, \dots, f^{-1}_{k_K,j_K}$ for these branches. Since we only have light overlaps, without loss of generality we may assume that $\forall x \in Z$: $f^{-1}_{k_\beta,j_\beta}(x) < f^{-1}_{k_\gamma,j_\gamma}(x)$ if $\beta < \gamma$.

Let us define

$$\varepsilon := \max\left\{\varepsilon' > 0 \middle| \forall \beta \in [K-1], \forall A \subset \mathbb{R}, |A| = \varepsilon' : \right. \\ \left. f^{-1}_{k_{\beta+1},j_{\beta+1}}(A) \cap f^{-1}_{k_\beta,j_\beta}(A) = \emptyset\right\}. \tag{4.14}$$

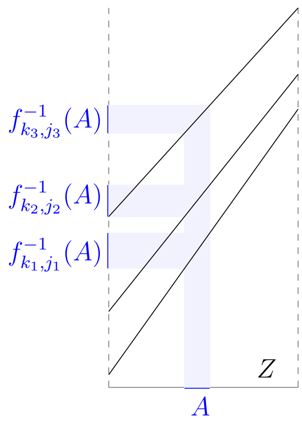

Figure 4.4: If the interval $A \subset Z$ is small enough, then its images by the three light overlapping branches are disjoint.

Since we only have light overlaps, ε is a well-defined positive number. The images of any interval $A \subset Z$ with length at most ε by the branches $f^{-1}_{k_1,j_1}, \dots, f^{-1}_{k_K,j_K}$ must be disjoint. It is illustrated on Figure 4.4.

Let $\mathcal{Y}_Z$ be a finite partition of Z whose elements are all have length at most ε. By substituting $\mathcal{Y}_Z$ in the place of Z in $\mathcal{Z}_0$, we obtain a finite refinement $\mathcal{Y}$ of $\mathcal{Z}_0$. By (4.14), there are no multiple edges in the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$.

Assume now that light overlaps occur above $q > 1$ many monotonicity intervals $Z_1, \dots Z_q \in \mathcal{Z}_0$. For each $i \in [q]$, let ε_i be the number defined in (4.14) using the branches above Z_i, and let $\mathcal{Y}_{Z_i}$ be a finite partition of Z_i whose elements are all have length at most ε_i. By replacing Z_i in $\mathcal{Z}_0$ with $\mathcal{Y}_{Z_i}$ for every $i \in [q]$, we obtain the finite partition $\mathcal{Y}$. The Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$ does not contain any multiple edges. □

Lemma 4.3.4 implies that for a CPLIFS with only light overlaps [11, Corollary 1/i] also holds.

Proposition 4.3.5. *Let $\mathcal{F}$ be a CPLIFS with only light overlaps, and for a finite partition $\mathcal{Y}$ let $(\mathcal{D}(\mathcal{Y}), \to)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$. Then there exists a $\mathcal{Y}$ finite partition such that*

$$\lim_{N\to\infty} \varrho\left(\mathbf{F}_{\mathcal{D}(\mathcal{Y})\setminus\mathcal{D}(\mathcal{Y})_N}(s)\right) = 1, \tag{4.15}$$

where $\mathbf{F}(s)$ is the matrix associated to $(\mathcal{D}(\mathcal{Y}), \to)$.

For the convenience of the reader, we include here a modified version of the proof of [11, Corollary 1/i].

Proof. According to Lemma 4.3.4, there exists a $\mathcal{Y}$ finite refinement of $\mathcal{Z}_0$ such that there are no multiple edges in $(\mathcal{D}, \to) := (\mathcal{D}(\mathcal{Y}), \to)$. Fix $N > 1$, and let $Z \in \mathcal{D} \setminus \mathcal{D}_N$. Further, let (k_1, j_1) be the label of one of the edges from Z, and let $(k_1, j_1), \ldots, (k_q, j_q)$ be a sequence of labels corresponding to a path of directed edges in $\mathcal{D} \setminus \mathcal{D}_N$ for some arbitrary $q > 0$. We will show that if $q \leq N$, then $(k_1, j_1), \ldots, (k_q, j_q)$ defines at most two directed paths of the form $Z_0 = Z \to Z_1 \to \cdots \to Z_q$ in $\mathcal{D} \setminus \mathcal{D}_N$.

Assume that $q < N$ and one of the endpoints of Z is a critical point. Without loss of generality suppose that $Z = [w, x]$ where $w \in \mathcal{K}$. The successors of Z by the branch (k_1, j_1) can only end in $f_{k_1,j_1}^{-1}(w), f_{k_1,j_1}^{-1}(x)$ or at some critical point. Out of them only at most one is in $\mathcal{D} \setminus \mathcal{D}_N$, since intervals of the form $[a, b]$ where $a \in T^{i_1}v_1, b \in T^{i_2}v_2,\ v_1, v_2 \in \mathcal{K}, 0 \leq i_1, i_2 \leq N$ are all contained in $\mathcal{D}_N$. Namely, the interval which ends in $f_{k_1,j_1}^{-1}(x)$. Therefore Z_1 is uniquely defined. Similarly, Z_i must be that sucessor of Z_{i-1} which ends in $f_{k_i,j_i}^{-1} \circ \cdots \circ f_{k_1,j_1}^{-1}(x)$, for $i \in [q]$. So in this case $Z_1, \ldots, Z_q$ are uniquely defined.

If none of the endpoints of $Z = [x, y]$ is a critical point, then there are at most two successors of Z in $\mathcal{D} \setminus \mathcal{D}_N$. Both of these intervals end in a critical point, so we can apply the previous argument for them. Thus we have two versions for $Z_1, \ldots, Z_q$.

We just showed that in the matrix $\left(\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)\right)^{nN}$, in the row of an arbitrary $Z \in \mathcal{D} \setminus \mathcal{D}_N$ there are at most $2^n \cdot K^N$ many non-zero elements for all $n > 0$. Here we used K^N as an upper bound for the possible number of N length paths in $\mathcal{D} \setminus \mathcal{D}_N$. It follows from Lemma 4.3.4 that the elements of $\left(\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)\right)^{nN}$ are all bounded from above by 1, since there are no multiple edges in $(\mathcal{D}, \to)$. Thus

$$\varrho\left(\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)\right) \leq \sqrt[nN]{\| \left(\mathbf{F}_{\mathcal{D}\setminus\mathcal{D}_N}(s)\right)^{nN} \|_\infty} \leq \sqrt[nN]{2^n K^N} = \sqrt[N]{2} \cdot \sqrt[n]{K},$$

for any $n \geq 1$, and with this the statement is proved. □

Lemma 4.3.2 and Proposition 4.3.5 together gives

$$\mathcal{F} \text{ has only light overlaps} \implies \mathcal{F} \text{ is limit-irreducible.}$$

That is for a CPLIFS $\mathcal{F}$ with only light overlapsand with a generated self-similar system satisfying the ESC, we always have $\dim_{\mathrm{H}} \Lambda = \min\{1, s_{\mathcal{F}}\}$, where Λ is the attractor of $\mathcal{F}$.

4.3.2 The case of cross overlaps

We call the elements of the set

$$\left\{x \in \mathcal{I} \middle| \exists k_1, k_2 \in [m], \exists j_1 \in [l(k_1)], \exists j_2 \in [l(k_2)], k_1 \neq k_2 : f_{k_1,j_1}^{-1}(x) = f_{k_2,j_2}^{-1}(x)\right\}$$

intersecting points. They form a subset of the critical points $\mathcal{K}$. Let $w \in I$ be an intersecting point, then the elements of $\mathcal{D}$ can only contain w as their endpoint. If $D \in \mathcal{D}$ ends in w, then we say that D **is causing cross overlaps** at w.

Lemma 4.3.6. *Let $\mathcal{F}$ be a CPLIFS with associated expanding multi-valued mapping T. Let $x_0 \in I$ be an intersecting point. If the generated self-similar system $\mathcal{S}$ of $\mathcal{F}$ satisfies the ESC, then there is no finite N for which $x_0 \in T^N(x_0)$.*

Proof. We will prove the statement by contradiction and assume that there is a finite $N > 0$ such that $x_0 \in T^N(x_0)$. Let $f^{-1}_{k'_1,j'_1}$ and $f^{-1}_{\widehat{k}_1,\widehat{j}_1}$ be two different branches of T that map x_0 to the same value. These must exist since x_0 is an intersecting point. Without loss of generality, assume that the sequence of branches $((k_1, j_1), \dots, (k_N, j_N))$ maps x_0 to itself. Precisely,

$$f^{-1}_{k_N,j_N} \circ \cdots \circ f^{-1}_{k_1,j_1}(x_0) = x_0.$$

The same holds for the sequence of branches $((k_1, j_1), \dots, (k_N, j_N), (k'_1, j'_1))$ and $\Big((k_1, j_1), \dots, (k_N, j_N), (\widehat{k}_1, \widehat{j}_1)\Big)$ as well.

For a given branch $f^{-1}_{k,j}$, we write $S_{k,j}$ for the corresponding element of the generated self-similar IFS $\mathcal{S}$. It follows that

$$S_{(k'_1,j'_1),(k_1,j_1),\dots,(k_N,j_N)}(x) = S_{(\widehat{k}_1,\widehat{j}_1),(k_1,j_1),\dots,(k_N,j_N)}(x).$$

Using these two functions, we can construct at least two identical iterates with different codes for any level $n > N$. It implies that the ESC fails for $\mathcal{S}$. □

Lemma 4.3.7. *Let $\mathcal{F}$ be a CPLIFS whose generated self-similar system satisfies the ESC. Let T be the expanding multi-valued mapping associated to $\mathcal{F}$ and W be the set of all intersecting points. Fix $P > 0$. Then there exists a finite refinement $\mathcal{Y}$ of $\mathcal{Z}_0$ such that*

$$\forall Z \in \mathcal{Y}, \forall w \in W, \forall n \in [P] : w \in Z \implies Z \cap (\cup T^n(Z)) = \emptyset. \tag{4.16}$$

Proof. Let $w \in W$ be an arbitrary intersecting point. According to Lemma 4.3.6, $\forall n \in [P] : w \notin T^n(w)$. That is, the distance of w and the set $\cup_{n\in[P]} T^n(w)$ is positive. Let $d > 0$ be this distance. Recall that $\rho_{\min}$ denotes the smallest contraction ratio in $\mathcal{F}$, hence $1/\rho_{\min}$ is the largest slope of T.

Let $p \in \mathbb{R}$ be a point that satisfies

$$|w - p| < \frac{d\rho^P_{\min}}{1 + \rho^P_{\min}}. \tag{4.17}$$

Let $Z \in \mathcal{Z}_0$ be a monotonicity interval that contains w, and let p be the only point in Z that satisfies (4.17). Intersecting points are also critical points, thus Z can only contain one such point. We cut Z into two closed intervals by p and call them Y_Z, Y'_Z.

We can construct the pair of intervals Y_Z, Y'_Z for any monotonicity interval $Z \in \mathcal{Z}_0$ that contains an intersecting point $w \in W$. By replacing all Z in $\mathcal{Z}_0$ that causes cross overlaps with the correspoding $\{Y_Z, Y'_Z\}$, we obtain a finite partition $\mathcal{Y}$ that satisfies (4.16). □

Proof of Proposition 4.3.1. We are going to construct a limit-irreducible partition of $\mathcal{F}$ with the help of Lemma 4.3.7 and Lemma 4.3.4. According to Definition 4.1.9, we may restrict ourselves to $s \in (0, \dim_{\mathrm{H}} \Lambda)$. Then, $s < s_{\mathcal{F}}$ by Corollary 1.0.1. It follows that $\varrho(\mathbf{F}(s)) > 1$, so we can fix an $\varepsilon > 0$ for which $\varrho(\mathbf{F}(s)) > 1 + \varepsilon$.

Write M for the number of intersection points in the system and K for the order of overlapping. Fix a $P > 0$ big enough such that

$$\sqrt[P]{K} < \sqrt[M+1]{1+\varepsilon}. \tag{4.18}$$

We apply Lemma 4.3.7 to $\mathcal{F}$ and P to obtain the finite partition $\mathcal{Y}$. Let $(\mathcal{D}, \rightarrow)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$, and let $Z \in \mathcal{Y}$ be an interval that causes a cross overlapping. Thanks to the construction of $\mathcal{Y}$, Z does not intersect with its first P successors. In other words, no P length directed path in $\mathcal{D} \setminus \mathcal{D}_P$ can visit Z more than once.

Let Z_{cross} be the set of all images of Z by the different cross overlapping branches defined above it. The elements of Z_{cross} are nested. Therefore, using the cross overlapping branch with the biggest expansion ratio, we can dominate every directed path of length at most P in $\mathcal{D} \setminus \mathcal{D}_P$ that goes through Z and contains an element of Z_{cross}. This means that for every $n \in [P]$ and for every directed path $\widehat{Z} \rightarrow C_1 \rightarrow \cdots \rightarrow C_n$ in $\mathcal{D} \setminus \mathcal{D}_P$ with $\widehat{Z} \subset Z$ and $C_1 \in Z_{\text{cross}}$, there exists a directed path $Z \rightarrow D_1 \rightarrow \cdots \rightarrow D_n$ in $\mathcal{D} \setminus \mathcal{D}_P$ such that D_1 is the successor of Z by a branch of the biggest slope, and $\forall k \in [n] : C_k \subset D_k$. It is essentially the same as erasing all other cross overlapping branches of T above Z, see Figure 4.5. We do the same domination for all $Z \in \mathcal{Y}$ that causes a cross overlapping. Let $\mathbf{F}^{\max}(s)$ be the matrix of this dominated system. We write max in the upper index to indicate that after the domination the only cross overlapping branch left above each Z that originally caused a cross overlap is the one with the biggest expansion ratio.

Our new system, the one we obtained by dominating the cross overlapping branches, can only have light overlaps. The Markov diagram of this system

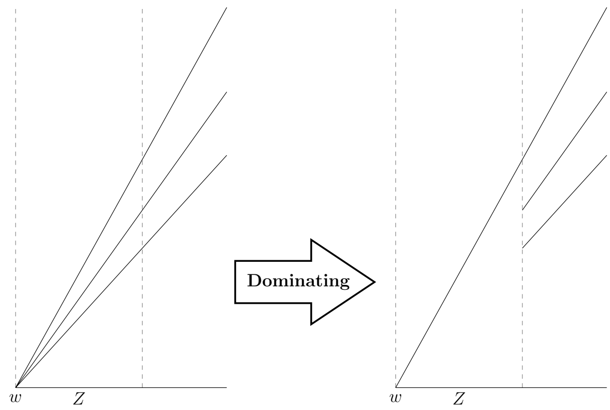

Figure 4.5: This figure illustrates how we handle cross overlappings by dominating the branches around the intersecting point w with a branch of the biggest slope. The nodes of the Markov diagram remain the same, we only delete some edges.

has the same nodes as the original, we only erased edges by the domination. Using Proposition 4.3.5, we get the finite refinement $\mathcal{Y}'$ of $\mathcal{Y}$ for which (4.15) holds with $\mathbf{F}^{\max}(s)$. That is there exists an $N > P$ such that

$$\varrho\left(\mathbf{F}^{\max}_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)\right) < 1+\varepsilon, \tag{4.19}$$

where $(\mathcal{D}', \to)$ is the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}'$. Let $\mathbf{F}'(s)$ be the matrix associated to $(\mathcal{D}', \to)$. It follows from the construction of the matrix $\mathbf{F}^{\max}(s)$ that every entry of $\mathbf{F}^{\max}(s)$ is smaller or equal to the corresponding entry of $\mathbf{F}'(s)$.

Now we show that the submatrix $\mathbf{F}'_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)$ has spectral radius smaller than that of $\mathbf{F}'(s)$. Let $Z \in \mathcal{Y}$ be one of those intervals that caused a cross overlapping before the domination. Observe that in $(\mathcal{D}' \setminus \mathcal{D}'_N, \to)$ at most K many directed edges start from Z. That is we dominated at most K^M many paths in $(\mathcal{D}' \setminus \mathcal{D}'_N, \to)$ with a single one. By this we obtain the upper bound

$$\left(\| \left(\mathbf{F}'_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)\right)^{nP} \|_\infty\right)^{\frac{1}{nP}} \le \left(\sqrt[P]{K}\right)^M \left(\| \left(\mathbf{F}^{\max}_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)\right)^{nP} \|_\infty\right)^{\frac{1}{nP}},$$

for any $1 \le n$. It follows that

$$\varrho\left(\mathbf{F}'_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)\right) \le \left(\sqrt[P]{K}\right)^M \cdot \varrho\left(\mathbf{F}^{\max}_{\mathcal{D}'\setminus\mathcal{D}'_N}\right). \tag{4.20}$$

We conclude the proof by substituting (4.19) and (4.18) into (4.20)

$$\varrho\left(\mathbf{F}'_{\mathcal{D}'\setminus\mathcal{D}'_N}(s)\right) < 1+\varepsilon < \varrho\left(\mathbf{F}'(s)\right).$$

According to Theorem 4.3.2, $\mathcal{F}$ is limit-irreducible. $\square$

This proof suggests stronger description of limit-irreducibility. Namely, the proof of Lemma 4.3.7 shows that if the CPLIFS $\mathcal{F}$ is not limit-irreducible, then there must exist a crossing point b with a periodic orbit. This observation leads us to the next lemma.

Lemma 4.3.8. *Let $\mathcal{F}$ be a CPLIFS with generated self-similar IFS $\mathcal{S}$. If $\mathcal{S}$ has no exact overlappings then $\mathcal{F}$ is limit-irreducible.*

Proof. Let $b \in \mathcal{K}$ be a crossing point, and let $f_{k,j_l}^{-1}, f_{k,j_r}^{-1}$ be two branches for which $f_{k,j_l}^{-1}(b) = f_{k,j_r}^{-1}(b)$. Arguing by contradiction, it is enough to show that the existence of a path $b \to_{k,j_l} \to \cdots \to b$ in $(\mathcal{G}, \to)$ would imply exact overlappings in $\mathcal{S}$.

Let $n \geq 0$, and assume that for $(k_n, j_n) \dots (k_1, j_1)$ we have

$$f_{(k_n,j_n)\dots(k_1,j_1)(k,j_l)}^{-1}(b) = b.$$

Writing $\bar{\imath} = (k_1, j_1) \dots (k_n, j_n)$, it follows that

$$S_{k,j_r} \circ S_{\bar{\imath}} \circ S_{k,j_l} \circ S_{\bar{\imath}} \equiv S_{k,j_l} \circ S_{\bar{\imath}} \circ S_{k,j_r} \circ S_{\bar{\imath}},$$

as these two similarities have the same slope and they take the same value at b. That is the existence of such a word $\bar{\imath}$ indeed implies exact overlaps in $\mathcal{S}$. $\square$

Remark 4.3.9. *The proof of Lemma 4.3.8 implies that limit-irreducibility is connected to the crossing points of the IFS. If none of the crossing points has a periodic orbit, then the Markov diagram of $\mathcal{F}$ is limit-irreducible.*

Chapter 5

Graph-directed representations

It is important to note that not every continuous piecewise linear iterated function system can be represented as a self-similar GDIFS. We have to show that the monotonicity partition of our CPLIFS admits a Markov structure, otherwise the theory of GDIFSs cannot be applied. In particular, we must have a finite Markov diagram.

A breaking point on the attractor with aperiodic coding immediately implies that the Markov diagram is infinitely big. We have seen in Section 3.1, that regular CPLIFSs always have an associated self-similar graph-directed IFS. In this section we cover the remaining third case of CPLIFSs that may have some breaking points on the attractor, but only with periodic coding [19].

5.1 Breaking points with periodic coding

Throuhout this section we will work only with CPLIFS $\mathcal{F}$ that satisfies the following assumption.

Assumption 1. *If a breaking point b of a function in $\mathcal{F}$ falls onto the attractor $\Lambda^{\mathcal{F}}$, then it only has periodic codings in the symbolic space. Precisely,*

$$\exists \mathbf{i} \in \Sigma : \Pi(\mathbf{i}) = b \implies \mathbf{i} \textit{ is periodic}, \tag{5.1}$$

where $\Pi : \Sigma \to \Lambda$ denotes the natural projection defined by $\mathcal{F}$.

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS that satisfies Assumption 1. Let $b_1, \dots b_Q$ be those breaking points of $\mathcal{F}$ that fall onto the attractor $\Lambda := \Lambda^{\mathcal{F}}$, and let $\mathbf{i}_1, \dots, \mathbf{i}_{Q'} \in \Sigma$ be all the words that satisfy

$$\forall k \in [Q'], \exists j \in [Q] : f_{\mathbf{i}_k} b_j = b_j.$$

Since there are only finitely many functions in $\mathcal{F}$, and they are all piecewise linear consisting of finitely many affine parts, $Q^{'}$ must be a finite number.

According to (5.1), $b_1, \dots b_Q$ can only have periodic codes. By definition, we can obtain these codes by concatenating some words $\mathbf{i}_1, \dots, \mathbf{i}_{Q'}$. Note that, as we have no separation condition on $\mathcal{F}$, some breaking points might have multiple codes, hence $Q \leq Q^{'}$. Further, we write P for the smallest common multiplier of the numbers $|\mathbf{i}_1|, \dots, |\mathbf{i}_{Q'}|$.

Now we have all the necessary notations to associate a self-similar GDIFS to $\mathcal{F}$. Consider the cylinders of level P:

$$\Lambda^P := \left\{ f_{\mathbf{j}}(\Lambda) : \mathbf{j} \in \Sigma^P \right\}.$$

For a $\mathbf{j} \in \Sigma^P$, we call $\phi_C \in \mathbb{R}$ the fixed point of the set $C \in \Lambda^P$ if $C = f_{\mathbf{j}}(\Lambda)$ and $f_{\mathbf{j}}(\phi_C) = \phi_C$.

We construct the graph-directed sets from the elements of Λ_P in the following way:

1. If $C \in \Lambda^P$ does not contain any of the the points $b_1, \dots, b_Q$, then C is a graph-directed set;

2. If $C = [c_l, c_r] \in \Lambda^P$ contains a breaking point as an inner point, then we cut C into two new closed sets $C^- := [c_l, \phi_C]$, $C^+ := [\phi_C, c_r]$ by its fixed point ϕ_C. The sets C^-, C^+ are graph-directed sets.

That is, we can define the set G of graph-directed sets in the following way.

$$\begin{aligned} \forall C \in \Lambda^P : \forall j \in [Q] : b_j \notin C &\implies C \in G \\ \exists j \in [Q] : b_j \in C &\implies C^-, C^+ \in G. \end{aligned}$$

We say that $\mathbf{i}_C \in \Sigma^P$ is the code of the set $C \in \Lambda^P$ if $\Pi(\mathbf{i}_C) = C$. This way we can define the code of graph directed sets as well. In particular, if $C^{'} \in G \setminus \Lambda_P$, then $\mathbf{i}_{C'} := \mathbf{i}_C$, where $C \in \Lambda_P$ and $C^{'} \in \{C^-, C^+\}$. Note that the graph directed sets $C_1, C_2 \in G$ will share the same code if $C_1 = C_-$ and $C_2 = C_+$ for some $C \in \Lambda^P$.

Lemma 5.1.1. *The elements of G do not contain any breaking point as an inner point.*

Proof. As $\mathcal{F}$ satisfies Assumption 1 and P is the smallest common multiplier of the numbers $|\mathbf{i}_1|, \dots, |\mathbf{i}_{Q'}|$, if b_j is contained in $C \in \Lambda^P$ for some $j \in [Q]$, then $b_j = \phi_C$.

Since we cut the corresponding elements of Λ_P into two by their fixed points, it follows that the elements of G can only contain a breaking point b_j as a boundary point. □

To associate a GDIFS to $\mathcal{F}$, we need a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. We already defined the graph-directed sets as the elements of G. Accordingly, we define the set of nodes as $\mathcal{V} = \{1, \ldots, |G|\}$. For an arbitrary graph directed set $C \in G$, let $\mathbf{i}_C \in \Sigma^P$ be its code and $q_C \in \mathcal{V}$ be the node in the graph representing this set. The set of edges $\mathcal{E}$ is defined the following way

$$C, C^{'} \in G, \text{ and } f_{\mathbf{i}_C}(C^{'}) \in C \implies (q_C, q_{C'}) \in \mathcal{E}.$$

For an edge $e = (q_C, q_{C'}) \in \mathcal{E}$, we define the corresponding contraction as $f_e = f_{\mathbf{i}_C}$. We call the graph-directed system defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\{f_e\}_{e \in \mathcal{E}}$ the **associated graph-directed system of** $\mathcal{F}$, and we denote it by $\mathcal{F}_{\mathcal{G}}$. According to Lemma 5.1.1, $\mathcal{F}_{\mathcal{G}}$ is always self-similar. We just obtained the following result.

Theorem 5.1.2. *Let $\mathcal{F}$ be a CPLIFS with attractor Λ. If $\mathcal{F}$ satisfies Assumption 1, then Λ is the attractor of a self-similar graph directed iterated function system.*

5.2 Fixed points as breaking points

In general, we cannot give a formula for the Hausdorff dimension of the attractor of a CPLIFS, but we can in some special cases, using the previously described construction. Here we demonstrate it on the case of CPLIFS $\mathcal{F}$ with the following properties:

1. $\mathcal{F}$ is injective,
2. the functions of $\mathcal{F}$ have positive slopes,
3. the functions of $\mathcal{F}$ can only break at their fixed points,
4. the first cylinder intervals are disjoint.

We construct the associated directed graph, and then we give a recursive formula for the Hausdorff dimension of these systems.

Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be an injective CPLIFS, and let Λ be its attractor. For each $i \in [m]$, we denote the fixed point of f_i with ϕ_i. We further assume that the only breaking point of f_i is ϕ_i, for every $i \in [m]$. We call f_1 and f_m the maps with the smallest and largest fixed points respectively. Then the interval I that supports the attractor is defined by ϕ_1 and ϕ_m. From now on, without loss of generality we suppose that $\phi_1 = 0$ and $\phi_m = 1$.

Let $m = 3$, and define the functions of $\mathcal{F}$ as follows.

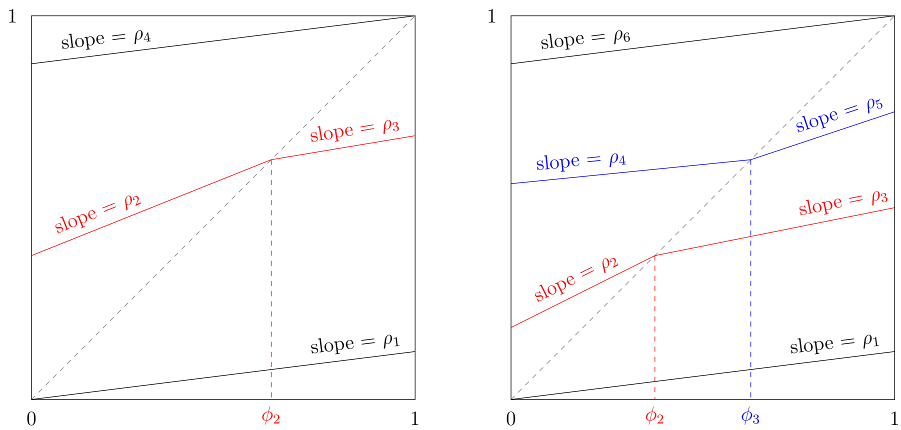

Figure 5.1: Illustration of CPLIFSs that we discuss in this section with $m = 3$ and 4 functions.

$$f_1(x) = \rho_1 x, \qquad f_2(x) = \begin{cases} \rho_2 x + \phi_2(1-\rho_2), \text{ if } x \in [0, \phi_2] \\ \rho_3 x + \phi_2(1-\rho_3), \text{ if } x \in [\phi_2, 1] \end{cases}$$
$$f_3(x) = \rho_4 x + (1-\rho_4)$$

We require here that $\rho_2 \neq \rho_3$ and that $\mathcal{F}$ satisfies the first cylinder intervals of the system are disjoint. Clearly, f_2 only breaks at its fixed point ϕ_2. Thanks to this property, all elements of the generated self-similar IFS are self-mappings of certain intervals. Namely, $S_{2,1}$ is a self-map of $[0, \phi_2]$ and $S_{2,2}$ is a self-map of $[\phi_2, 1]$. It implies that we can associate a graph-directed function system with directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = [4]$ and the edges are defined by the incidence matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

As we did in 2.11, we define the following matrix

$$\begin{bmatrix} \rho_1^s & \rho_1^s & \rho_1^s & \rho_1^s \\ \rho_2^s & \rho_2^s & 0 & 0 \\ 0 & 0 & \rho_3^s & \rho_3^s \\ \rho_4^s & \rho_4^s & \rho_4^s & \rho_4^s \end{bmatrix}.$$

With the help of Theorem 2.3.9 and the Perron Frobenius Theorem, the Hausdorff dimension of Λ equals to the solution of the following equation

$$
\begin{aligned}
0 = \det U_4^{(s)} &:= \det \begin{bmatrix} \rho_1^s - 1 & \rho_1^s & \rho_1^s & \rho_1^s \\ \rho_2^s & \rho_2^s - 1 & 0 & 0 \\ 0 & 0 & \rho_3^s - 1 & \rho_3^s \\ \rho_4^s & \rho_4^s & \rho_4^s & \rho_4^s - 1 \end{bmatrix} \\
&= 1 - \rho_1^s - \rho_2^s - \rho_3^s - \rho_4^s + \rho_1^s\rho_3^s + \rho_2^s\rho_3^s + \rho_2^s\rho_4^s.
\end{aligned}
$$

Thus the Hausdorff dimension of $\mathcal{F}$ is the unique number s that satisfies the equation

$$
V_4(s) := 1 - \rho_1^s - \rho_2^s - \rho_3^s - \rho_4^s + \rho_1^s\rho_3^s + \rho_2^s\rho_3^s + \rho_2^s\rho_4^s = 0. \tag{5.2}
$$

We call $V_4(s)$ the **determinant function** of $\mathcal{F}$. It is easy to check that (5.2) gives back the similarity dimension in the self-similar case ($\rho_2 = \rho_3$), thus it is a consistent extension of the dimension theory of self-similar systems.

In a similar fashion, we write $V_{2m-2}(s)$ for the determinant function of a CPLIFS with $m \geq 3$ functions and $U_{2m-2}^{(s)}$ for the corresponding matrix (the matrix of the associated GDIFS minus the appropriate dimensional identity matrix), to keep track of the number of different slopes as parameters in the notation. For instance, if $m = 4$, then

$$
U_{2m-2}^{(s)} = U_6^{(s)} = \begin{bmatrix} \rho_1^s - 1 & \rho_1^s & \rho_1^s & \rho_1^s & \rho_1^s & \rho_1^s \\ \rho_2^s & \rho_2^s - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_3^s - 1 & \rho_3^s & \rho_3^s & \rho_3^s \\ \rho_4^s & \rho_4^s & \rho_4^s & \rho_4^s - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_5^s - 1 & \rho_5^s \\ \rho_6^s & \rho_6^s & \rho_6^s & \rho_6^s & \rho_6^s & \rho_6^s - 1 \end{bmatrix}.
$$

Fixing the slopes $\rho_1, \dots, \rho_{2m-2}$ let us express $V_{2m-2}(s)$ recursively, since this way the determinant of the upper left $2n \times 2n$ block in $U_{2m-2}^{(s)}$ equals to $U_{2n}(s)$ for each $0 < n < m - 1$. After expanding the determinant of $U_{2m-2}^{(s)}$ by the second row from below we obtain the following formula

$$
V_{2m-2}(s) = (1 - \rho_{2m-2}^s - \rho_{2m-3}^s)V_{2m-4}(s) + \rho_{2m-2}^s \sum_{i=1}^{2m-4} (-1)^i V_{2m-4,i}(s), \tag{5.3}
$$

where $V_{2m-4,i}(s)$ is the determinant of the matrix obtained by erasing the i-th column of $U_{2m-4}^{(s)}$ and then adding the first $(2m - 4)$ elements of the last column of $U_{2m-2}^{(s)}$ as a column vector from the right. For example, $V_{4,2}$ is the determinant of the following matrix

$$\begin{bmatrix} \rho_1^s - 1 & \rho_1^s & \rho_1^s & \rho_1^s \\ \rho_2^s & 0 & 0 & 0 \\ 0 & \rho_3^s - 1 & \rho_3^s & \rho_3^s \\ \rho_4^s & \rho_4^s & \rho_4^s - 1 & 0 \end{bmatrix}.$$

If we calculate $V_{2m-2,i}(s)$ by expanding the corresponding determinant by the second from the last row, it is easy to see that depending on $i \in [2m-2]$ we obtain the following values

$$V_{2m-2,i}(s) = \\ = \begin{cases} (1-\rho_{2m-2}^s)V_{2m-4,i}(s), & i \in [2m-4] \\ \rho_{2m-3}^s(1-\rho_{2m-2}^s)V_{2m-4}(s), & i = 2m-3 \\ \rho_{2m-2}^s(\sum_{j=1}^{2m-4}(-1)^j V_{2m-4,j}(s) - \rho_{2m-3}^s V_{2m-4}(s)), & i = 2m-2. \end{cases} \tag{5.4}$$

By using (5.3) and (5.4) we can construct $V_{2m-2}(s)$ for any $m \geq 3$. That is, with this recursive algorithm we can calculate the Hausdorff dimension of any CPLIFS that satisfies the IOSC if its functions only break at their respective fixed points.

Chapter 6

Continuity of the natural dimension

The main goal of this chapter is to prove results on the dependency of the natural pressure on the parameters of a CPLIFS. We managed to show that the natural dimension of a CPLIFS changes continuously if its generated self-similar IFS has no exact overlappings [17].

We want to show that the natural dimension of a given CPLIFS does not change too much if we perturb its parameters. To do this, we need to define what we mean by two CPLIFSs being close to each other.

Definition 6.0.1. *Let $[c,d], [\widehat{c}, \widehat{d}] \subset \mathbb{R}$ be two closed intervals. We say that they are* ***ε-close*** *if*

$$|c - \widehat{c}| < \varepsilon \text{ and } |d - \widehat{d}| < \varepsilon.$$

We say that the finite partitions $\mathcal{Y} = \{Y_1, \dots, Y_N\}$ and $\widehat{\mathcal{Y}} = \{\widehat{Y}_1, \dots, \widehat{Y}_N\}$ are ***ε-close*** *to each other if for all $j \in [N]$, Y_j and $\widehat{Y}_j$ are ε-close.*

Definition 6.0.2. *We say that the CPLIFSs $\mathcal{F} = \{f_k\}_{k=1}^m$ and $\widehat{\mathcal{F}} = \{\widehat{f}_k\}_{k=1}^m$ are* ***ε-close*** *if*

- **(a)** *their monotonicity partitions are ε-close,*
- **(b)** *$\forall k \in [m] : f_k$ and $\widehat{f}_k$ has the same number of breaking points,*
- **(c)** *$\forall k \in [m], \forall j \in [l(k)] : \left| \log |f'_{k,j}| - \log |\widehat{f}'_{k,j}| \right| < \varepsilon$,*
- **(d)** *$\forall k \in [m] : \|\widehat{f}_k - f_k\|_\infty < \varepsilon$.*

Theorem 6.0.3. *Let $\mathcal{F}$ be a CPLIFS with generated self-similar IFS $\mathcal{S}$. Suppose that $\mathcal{S}$ has no exact overlapping. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all CPLIFS $\widehat{\mathcal{F}}$ which is δ-close to $\mathcal{F}$*

$$|s_{\mathcal{F}} - s_{\widehat{\mathcal{F}}}| < \varepsilon,$$

where $s_{\mathcal{F}}$ and $s_{\widehat{\mathcal{F}}}$ are the natural dimensions of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively.

As an application of Theorem 6.0.3, in section 6.2 we prove that under mild conditions the Lebesgue measure of the attractor of a CPLIFS is typically positive if the natural dimension is strictly bigger than 1.

First, we give an example demonstrating that the natural dimension is not necessarily continuous with respect to the parameters of a CPLIFS.

Example 6.0.4. *Pick an arbitrary $\varepsilon > 0$, and let us define the following piecewise linear functions.*

$$f_1(x) = \begin{cases} \frac{2}{5}x, \ x < 0 \\ \frac{1}{5}x, \ x \geq 0 \end{cases}, \quad f_2(x) = \frac{1}{3}x, \quad \widehat{f}_2(x) = \frac{1}{3}x + \varepsilon$$

Consider the CPLIFSs $\mathcal{F} = \{f_1, f_2\}$ and $\widehat{\mathcal{F}} = \{f_1, \widehat{f}_2\}$. Write $\Lambda, \widehat{\Lambda}$ for the attractors and $\Phi(s), \widehat{\Phi}(s)$ for the natural pressure functions of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively. These two iterated function systems are Clearly ε-close to each other (see Definition 6.0.2).

It is easy to see that $\widehat{\Lambda}$ is a Cantor set. Since $\mathcal{F}$ satisfies the strong separation property, $\widehat{s}_{\text{nat}} = \dim_{\text{H}} \widehat{\Lambda}$ is the unique solution of

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{5}\right)^s = 1.$$

As 0 is the fixed point of both f_1 and f_2, the supporting interval I of $\mathcal{F}$ is $[-\frac{1}{2}, \frac{1}{2}]$. The slope of f_1 is strictly bigger than $\frac{1}{5}$ over $[-\frac{1}{2}, 0]$, so s_{nat} is expected to be strictly bigger than $\widehat{s}_{\text{nat}}$.

Set $\Sigma := \{1, 2\}^{\mathbb{N}}$, and let $\#_2(\bar{\imath})$ be the number of 2-s in $\bar{\imath} \in \Sigma$. Observe that for any $\bar{\imath} \in \Sigma$ the length of $I_{\bar{\imath}}$ only depends on $\#_2(\bar{\imath})$ and not on the position of the 2-s. A short calculation gives

$$|I_{\bar{\imath}}| = \left(\frac{1}{3}\right)^{n-k} \frac{2^k + 1}{2 \cdot 5^k}, \quad \text{if } \#_2(\bar{\imath}) = k.$$

The natural pressure function of $\widehat{\mathcal{F}}$ *is*

$$\widehat{\Phi}(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\bar{\imath}\in\Sigma: \\ |\bar{\imath}|=n}} |I_{\bar{\imath}}|^s = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{k=0}^{n} \sum_{\substack{\bar{\imath}\in\Sigma: \\ |\bar{\imath}|=n \text{ and } \#_2(\bar{\imath})=k}} |I_{\bar{\imath}}|^s =$$

$$= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{3}\right)^{s(n-k)} \left(\frac{2^k+1}{2\cdot 5^k}\right)^s > \log\left(\left(\frac{1}{3}\right)^s + \left(\frac{1}{5}\right)^s\right).$$

Thus, $\widehat{s}_{\mathrm{nat}} < s_{\mathrm{nat}}$ *independently of* ε*.*

6.1 Results on continuity

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS. Recall that the critical points of $\mathcal{F}$ were defined as

$$\mathcal{K} := \cup_{k=1}^m \{f_k(0), f_k(1)\} \bigcup \cup_{k=1}^m \cup_{j=1}^{l(k)} f_k(b_{k,j}) \bigcup$$
$$\left\{x \in \mathcal{I} \middle| \exists k_1, k_2 \in [m], \exists j_1 \in [l(k_1)], \exists j_2 \in [l(k_2)], k_1 \neq k_2 : f_{k_1,j_1}^{-1}(x) = f_{k_2,j_2}^{-1}(x)\right\}.$$

We call $\mathcal{K}^{\mathrm{int}} \subset \mathcal{K}$ the **set of inner critical points** if it contains all critical points of $\mathcal{F}$ that are interior points of $\mathcal{I} = \cup_{k=1}^m f_k(I)$.

For technical reasons, we will need to differentiate the endpoints of the monotonicity intervals. As neighbouring intervals share a common endpoint, we introduce a new topological space, where some points of the line are doubled. We repeat the definition of the doubled points topology from Section 2.2, but this time we use the general setting.

Let $\mathcal{Z}_0$ be the monotonicity partition of $\mathcal{F}$, and let $\mathcal{Y}$ be a finite refinement of $\mathcal{Z}_0$. Now write $I = [u, v]$ for the supporting interval of $\mathcal{F}$, and define

$$E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\},\ W := \left(\bigcup_{j=0}^{\infty} T^{-j}(E \setminus \{u, v\})\right) \setminus \{u, v\},$$

where $T^{-1}(A)$ is the preimage of the set $A \subset \mathbb{R}$ by the multi-valued mapping T. Observe that $E = \mathcal{K}$ for $\mathcal{Y} = \mathcal{Z}_0$, thus we can say that E takes the role of the set of critical points for finite refinements of $\mathcal{Z}_0$. We define $\mathbb{R}_{\mathcal{Y}} := \mathbb{R} \setminus W \bigcup \{x^-, x^+ : x \in W\}$, and set the order $y < x^- < x^+ < z$ if $y < x < z$ holds in $\mathbb{R}$.

In our new set $\mathbb{R}_{\mathcal{Y}}$, which is not a subset of $\mathbb{R}$, the endpoints of the elements of $\mathcal{Y}$ and their preimages by T are doubled. Now we define a metric on $\mathbb{R}_{\mathcal{Y}}$. Following [23], for $y, z \in \mathbb{R}_{\mathcal{Y}}$ with $y \leq z$ let

$$n(y, z) := \min_{k\in\mathbb{N}_0} \left\{ k \,\middle|\, \exists x \in \bigcup_{j=0}^{k} T^{-j}(E \setminus \{u, v\}) : y < x^+ \text{ and } x^- < z \right\}$$

if such a k exists, and set $n(y,z) := \infty$ otherwise. Further define

$$\mathrm{dist}(y,z) := \begin{cases} |\pi(y) - \pi(z)| + \frac{1}{n(y,z)+1}, & \text{if } n(y,z) < \infty \\ |\pi(y) - \pi(z)|, & \text{otherwise.} \end{cases}$$

This gives rise to a metric $\mathrm{dist}(y,z)$ on $\mathbb{R}_{\mathcal{Y}}$ which induces the order topology.

We define the projection $\pi : \mathbb{R}_{\mathcal{Y}} \to \mathbb{R}$ as

$$\pi(y) = x \text{ if either } y = x \in \mathbb{R} \text{ or } y \in \{x^-, x^+\}.$$

This mapping not just connects $\mathbb{R}_{\mathcal{Y}}$ to $\mathbb{R}$, but also preserves the ordering

$$y, z \in \mathbb{R}_{\mathcal{Y}} : y < z \implies \pi(y) < \pi(z), \text{ or } y = x^-, z = x^+ \text{ for } x \in W.$$

Later it will be useful to jump from one doubled copy to another, so we define the function $\zeta : \pi^{-1}(W) \to \pi^{-1}(W)$

$$x \in W : \zeta(x^+) = x^-, \zeta(x^-) = x^+.$$

For convenience, we extend this map to $\mathbb{R}_{\mathcal{Y}}$ by setting

$$\zeta(x) = x \text{ for } x \in \mathbb{R}_{\mathcal{Y}} \setminus \pi^{-1}(W).$$

From now on, we will work on the topological space $\mathbb{R}_{\mathcal{Y}}$ endowed with the order topology. So far we defined every set on $\mathbb{R}$, hence we need to define their counterparts on $\mathbb{R}_{\mathcal{Y}}$. Let $\mathcal{I}_{\mathcal{Y}}$ be the closure of $\mathcal{I} \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Observe that $\mathcal{I}_{\mathcal{Y}}$ is compact. We define

$$E_{\mathcal{Y}} := \{x \in \mathbb{R}_{\mathcal{Y}} : \pi(x) \in E\} \bigcap \mathcal{I}_{\mathcal{Y}}.$$

We emphasize that taking the intersection with $\mathcal{I}_{\mathcal{Y}}$ is important, as we are only interested in the endpoints of the monotonicity intervals and not the endpoints of the possible gaps between them.

We write $N := |\mathcal{Y}|$ for the cardinality of $\mathcal{Y}$ and $E_{\mathcal{Y}} = \{a_1, \ldots, a_{2N}\}$ with $a_1 < \cdots < a_{2N}$. The multi-valued mapping $T|_{\mathcal{I} \setminus W \cup E}$ uniquely extends on $\mathbb{R}_{\mathcal{Y}}$ to a mapping $T_{\mathcal{Y}}$. Similarly, let $\mathcal{K}_{\mathcal{Y}}$ and $\mathcal{K}_{\mathcal{Y}}^{\mathrm{int}}$ denote the set of critical points and inner critical points of $T_{\mathcal{Y}}$ respectively. We will supress naming the partition in the lower indices when $\mathcal{Y} = \mathcal{Z}_0$.

Now we define a directed graph $(\mathcal{G}, \to)$, different from the Markov diagram, that describes the orbit of the elements of $E_{\mathcal{Y}}$. For $i \in [2N]$ let

$$\omega(i) := \{\bar{\imath} = ((k_1, j_1), \ldots, (k_n, j_n)) : f_{\bar{\imath}}^{-1}(a_i) \in \mathcal{I}_{\mathcal{Y}}\}.$$

Note that $\omega(i)$ contains the empty word $\emptyset$ for all $i \in [2N]$, as $a_i \in \mathcal{I}_{\mathcal{Y}}$ for all $a_i \in E_{\mathcal{Y}}$. If $\bar{\imath} \in \omega(i)$, then $\bar{\imath}|_j \in \omega(i)$ as well for all $j \leq |\bar{\imath}|$, where $\bar{\imath}|_j$ is the word consisting of the first j characters of $\bar{\imath}$. We define

$$\forall i \in [2N], \forall \bar{\imath} \in \omega(i) : a_{i,\bar{\imath}} := f_{\bar{\imath}}^{-1}(a_i).$$

Set $\mathcal{G} := \{a_{i,\bar{\imath}} : a_i \in \mathcal{K}_{\mathcal{Y}}, \bar{\imath} \in \omega(i)\}$. For $a, b \in \mathcal{G}$, there is an edge $a \rightarrow_{k,j} b$ in the graph $(\mathcal{G}, \rightarrow)$ if and only if $b = f_{k,j}^{-1}(a)$ or $\zeta(b) = f_{k,j}^{-1}(a)$. We write $a \rightarrow b$ if there exists an edge $a \rightarrow_{k,j} b$ for some $k \in [m]$ and $j \in [l(k)]$. Observe that this graph does not depend on the partition $\mathcal{Y}$. We call $(\mathcal{G}, \rightarrow)$ the **orbit graph of critical points** of $\mathcal{F}$.

Following Definition 4.1.6, we associate the matrix $\mathbf{G}(s)$, indexed by the elements of $\mathcal{G}$, to the graph $(\mathcal{G}, \rightarrow)$

$$[\mathbf{G}(s)]_{a,b} := \begin{cases} \sum_{(k,j): a \rightarrow_{(k,j)} b} \left|f_{k,j}'\right|^s, \text{ if } a \rightarrow b \\ 0, \text{ otherwise.} \end{cases} \tag{6.1}$$

As $(\mathbf{G}(s))^n$ is always bounded in the l^∞-norm, we can define

$$\varrho(\mathbf{G}(s)) := \lim_{n \to \infty} \|(\mathbf{G}(s))^n\|_\infty^{1/n}.$$

The following technical lemma will be useful later.

Lemma 6.1.1. *Let $\mathcal{F}$ be a CPLIFS with generated self-similar IFS $\mathcal{S}$ and orbit graph $(\mathcal{G}, \rightarrow)$. Let $\mathbf{F}(s)$ and $\mathbf{G}(s)$ be the matrices associated to the Markov diagram and the orbit graph respectively. If $\mathcal{S}$ does not have exact overlappings, then*

$$\varrho(\mathbf{G}(s)) \leq \varrho(\mathbf{F}(s)). \tag{6.2}$$

Proof. Fix a path of infinite length $d_1 \rightarrow d_2 \rightarrow \dots$ in $(\mathcal{G}, \rightarrow)$. Let $J \subset \mathbb{N}$ be the set of indices such that

$$\forall i \in J : \quad d_i \in \{\zeta(d) : d \in T_{\mathcal{Y}} d_{i-1}\}.$$

Note that $i \in J$ implies $d_i \in \mathcal{K}^{\text{int}}$. If $J = \emptyset$, then there exists a corresponding path in $(\mathcal{D}, \rightarrow)$. Namely, the path $D_1 \rightarrow D_2 \rightarrow \dots$, where for all $i \geq 1, d_i$ is an endpoint of D_i.

Clearly, (6.2) can fail only if there is a path in $(\mathcal{G}, \rightarrow)$ that has bigger weight than any constant multiple of the weight of any path in $(\mathcal{D}, \rightarrow)$. It can only happen if J is an infinite set. As $|\mathcal{K}^{\text{int}}| < \infty$, it is equivalent to a $d \in \mathcal{K}^{\text{int}}$ having a periodic orbit. However, a periodic orbit in general does not imply $\varrho(\mathbf{G}(s)) > \varrho(\mathbf{F}(s))$.

The elements of $\mathcal{K}^{\text{int}}$ fall into three categories: images of breaking points, crossing points, and endpoints of overlapping first cylinders. It is easy to see that $\varrho(\mathbf{G}(s)) > \varrho(\mathbf{F}(s))$ can only happen if the following is satisfied

$$\begin{aligned} &\exists d \in \mathcal{K}^{\text{int}}, \exists k \in [m], \exists j_l, j_r \in [l(k)+1] : \\ &\qquad f_{k,j_l}^{-1}(d) = f_{k,j_r}^{-1}(d), \text{ and } \exists\, d \to_{k,j_l} \cdots \to d \text{ path in } (\mathcal{G}, \to). \end{aligned} \tag{6.3}$$

That is we must have a $d \in \mathcal{K}^{\text{int}}$ which is an image of a breaking point and has a periodic orbit starting with the right branch.

Assume now that there exists a $d \in \mathcal{K}^{\text{int}}$ and branches $(k, j_l), (k, j_r)$ satisfying (6.3). It follows that for some $n \geq 0$

$$\exists \bar{\imath} = ((k_1, j_1), \ldots, (k_n, j_n)) : S_{\bar{\imath}(k,j_l)}(d) = d. \tag{6.4}$$

Since $\mathcal{S} = \{S_{k,j}\}_{k \in [m], j \in [l(k)]}$ is a self-similar IFS, (6.4) implies exact overlappings in the system. Namely, the functions $S_{\bar{\imath}(k,j_l)\bar{\imath}(k,j_r)}$ and $S_{\bar{\imath}(k,j_r)\bar{\imath}(k,j_l)}$ are identical.

□

6.1.1 Continuity of the Markov diagram

Let $\mathcal{F}$ be a CPLIFS with associated multi-valued mapping T and monotonicity partition $\mathcal{Z}_0$. Let $\mathcal{Y}$ be a finite refinement of $\mathcal{Z}_0$ and $(\mathcal{D}, \to)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$. We define

$$\mathcal{M} := \{(i, \bar{\imath}) : i \in [2N], \bar{\imath} \in \omega(i)\},$$

and $\mathcal{M}_r := \{(i, \bar{\imath}) \in \mathcal{M} : |\bar{\imath}| \leq r\}$ for $r \in \mathbb{N}_0$. P. Raith [23, p 108] proved that we can think of the Markov diagram as the image of $\mathcal{M}$ under a special mapping. While [23] only investigates the case when T is an expansive mapping of the line and not multi-valued, the mapping A constructed in the proof works in our case as well, since we differentiate between the images generated by the different branches.

Lemma 6.1.2. *There exists a mapping $A : \mathcal{M} \to \mathcal{D}$ with the following properties.*

(a) *$A(\mathcal{M}) = \mathcal{D}$ and $A(\mathcal{M}_r) = \mathcal{D}_r$ for all $r \in \mathbb{N}_0$.*

(b) *a_i is an endpoint of $A(i, \emptyset)$ for all $i \in [2N]$.*

(c) *$a_{i,\bar{\imath}}$ is an endpoint of $A(i, \bar{\imath})$ for all $(i, \bar{\imath}) \in \mathcal{M}$.*

(d) *Let $k \in [m]$ and $j \in [l(k)]$. We introduce a graph structure on $\mathcal{M}$ such that $c \to_{(k,j)} d$ in $\mathcal{M}$ implies $A(c) \to_{(k,j)} A(d)$ in $\mathcal{D}$. For every $c \in \mathcal{M}$ the map A is bijective from $\{d \in \mathcal{M} : c \to_{(k,j)} d\}$ to $\{D \in \mathcal{D} : A(c) \to_{(k,j)} D\}$.*

(e) *$c \in \mathcal{M}_r$ implies the existence of a $d \in \mathcal{M}_r$ with $A(c) \subset A(d)$ and either $A(c) = [a_d, a_c]$ or $A(c) = [a_c, a_d]$.*

This map A will be surjective, but need not be injective. A $C \in \mathcal{D}$ can be represented by multiple elements of $\mathcal{M}$. It implies that there might be several subsets in $\mathcal{M}$ whose image under A is $\mathcal{D}$. Each of them can take the role of the Markov diagram if they satisfy the following definition.

Definition 6.1.3. *$(\mathcal{A}, \to)$ is called a **variant of the Markov diagram** of $\mathcal{F}$ with respect to $\mathcal{Y}$ if $\mathcal{A} \subset \mathcal{M}$ satisfies the following properties for all $k \in [m]$ and $j \in [l(k)]$.*

(a) *For $i \in [2N]$ and $\bar{\imath} \in \omega(i)$, $(i, \bar{\imath}) \in \mathcal{A}$ implies $(i, \bar{\imath}|_j) \in \mathcal{A}$ for all $j \leq |\bar{\imath}|$.*

(b) *$c, d \in \mathcal{A}$ and $c \to_{(k,j)} d$ in $\mathcal{M}$ imply $c \to_{(k,j)} d$ in $\mathcal{A}$.*

(c) *$c, d \in \mathcal{A}$ and $c \to_{(k,j)} d$ in $\mathcal{A}$ imply either $c \to_{(k,j)} d$ in $\mathcal{M}$ or there exists a $d_0 \in \mathcal{M} \setminus \mathcal{A}$ with $c \to_{(k,j)} d_0$ and $A(d) = A(d_0)$.*

(d) *For $c \in \mathcal{A}$ the map $A : \{d \in \mathcal{A} : c \to_{(k,j)} d\} \to \{D \in \mathcal{D} : A(c) \to_{(k,j)} D\}$ is bijective.*

(e) *$A(\mathcal{A} \cap \mathcal{M}_r) = \mathcal{D}_r$ for $r \in \mathbb{N}_0$.*

Observe that $(\mathcal{M}, \to)$ and $(\mathcal{D}, \to)$ are also variants of the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$. For $r \in \mathbb{N}_0$ set $\mathcal{A}_r := \mathcal{A} \cap \mathcal{M}_r$.

Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram. We write $\mathbf{F}^{\mathcal{A}}(s)$ for the matrix associated to $(\mathcal{A}, \to)$, and define it analogously to 4.3. The following lemma is a straightforward consequence of Definition 6.1.3.

Lemma 6.1.4. *Let $\mathcal{F}$ be a CLPIFS, and let $\mathcal{Y}$ be a finite refinement of its monotonicity partition $\mathcal{Z}_0$. Let $(\mathcal{A}, \to), (\mathcal{A}', \to)$ be two variants of the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$. Then*

$$\varrho(\mathbf{F}^{\mathcal{A}}(s)) = \varrho(\mathbf{F}^{\mathcal{A}'}(s)).$$

Remark 6.1.5. *Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram, and fix $\varepsilon > 0$. Assume that there exists a finite irreducible $\mathcal{C} \subset \mathcal{D}$ such that $\varrho(\mathbf{F}_{\mathcal{C}}(s)) > \varrho(\mathbf{F}(s)) - \varepsilon$. Then obviously $\exists r : \mathcal{C} \subset \mathcal{D}_r$, and by part v) of Definition 6.1.3,*

$$\exists \widetilde{\mathcal{C}} \subset \mathcal{A}_r \text{ such that } \varrho(\mathbf{F}^{\mathcal{A}}_{\widetilde{\mathcal{C}}}(s)) > \varrho(\mathbf{F}(s)) - \varepsilon.$$

Raith proved [23, Lemma 6] that if two systems are close to each other, then the initial parts of their Markov diagrams coincide. Our systems has a more complicated structure due to the possible overlappings, but the construction given in his proof works here also.

Let $\mathcal{Y} = \{Y_1, \dots, Y_N\}$ be a finite partition. For $1 < i < N$, we say that $C \subset \mathbb{R}$ is $\mathcal{Y}$-close to Y_i if $C \subset Y_{i-1} \cup Y_i \cup Y_{i+1}$.

Lemma 6.1.6 ([23, Lemma 6]). *Let $\mathcal{F}$ be a CLPIFS, and let $\mathcal{Y}$ be a finite refinement of its monotonicity partition. Then for every $r \in \mathbb{N}$ there exists a $\delta > 0$ such that for every CPLIFS $\widehat{\mathcal{F}}$ which is δ-close to $\mathcal{F}$ and for every finite refinement $\widehat{\mathcal{Y}}$ of $\widehat{\mathcal{Z}_0}$ which is δ-close to $\mathcal{Y}$, there exists a variant $(\mathcal{A}, \to)$ of the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$ and a variant $(\widehat{\mathcal{A}}, \to)$ of the Markov diagram of $\widehat{\mathcal{F}}$ with respect to $\widehat{\mathcal{Y}}$ with the following properties.*

- **(i)** *$\widehat{\mathcal{A}}_r$ can be written as a disjoint union $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, such that $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_2$ are closed in $\widehat{\mathcal{A}}_r$ and $\widehat{\mathcal{A}}_0 \subset \mathcal{B}_0$ ($\mathcal{B}_1$ and $\mathcal{B}_2$ might be empty).*
- **(ii)** *Every $c \in \widehat{\mathcal{A}}_r$ has at most two successors in $\mathcal{B}_1 \cup \mathcal{B}_2$ by a given branch.*
- **(iii)** *There exists a bijective function $\phi : \mathcal{A}_r \to \mathcal{B}_0$, and there exists a function $\psi : \mathcal{B}_2 \to \mathcal{G}$.*
- **(iv)** *For $c, d \in \mathcal{A}_r$ and $k \in [m], j \in [l(k)]$ the property $c \to_{(k,j)} d$ in $\mathcal{A}$ is equivalent to $\phi(c) \to_{(k,j)} \phi(d)$ in $\widehat{\mathcal{A}}$. For $c, d \in \mathcal{B}_2$ the property $c \to_{(k,j)} d$ in $\widehat{\mathcal{A}}$ implies $\psi(c) \to_{(k,j)} \psi(d)$ in $\mathcal{G}$.*
- **(v)** *$A(c) = Y_j$ for $c \in \mathcal{A}_0$ implies $\phi(c) \in \widehat{\mathcal{A}}_0$ and $\widehat{A}(\phi(c)) = \widehat{Y}_j$.*
- **(vi)** *$c \in \mathcal{A}_r$ and $A(c) \subset A(d)$ for a $d \in \mathcal{A}_0$ imply $\widehat{A}(\phi(c)) \subset \widehat{A}(\phi(d))$. $c \in \mathcal{B}_2$ and $\psi(c) \in A(d)$ for a $d \in \mathcal{A}_0$ imply $\widehat{A}(c)$ is $\widehat{Y}$-close to $\widehat{A}(\phi(d))$.*
- **(vii)** *Let $\mathcal{P}$ be the set of all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\widehat{A}_r$ with $c_0 \in \widehat{\mathcal{A}}_0$, and set $\mathcal{N} := \{(d_0, \dots, d_r) : d_j \in \mathcal{A}_r \cup \mathcal{G} \text{ for } j \in \{0, \dots, r\}\}$. Then there exists a function $\chi : \mathcal{P} \to \mathcal{N}$.*
- **(viii)** *Let $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}, \chi(c_0 \to c_1 \to \cdots \to c_r) = (d_0, \dots, d_r)$ and $j \in \{0, 1 \dots, r\}$. $c_j \in \mathcal{B}_2$ is equivalent to $d_j \in \mathcal{G}$, and we have then $\psi(c_j) = d_j$. $c_j \in \mathcal{B}_0$ implies $\phi(d_j) = c_j$. $c_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ implies $\widehat{A}(c_j)$ is $\widehat{\mathcal{Y}}$-close to $\widehat{A}(\phi(d_j))$. Moreover, $c_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ implies $d_{j-1} \to d_j$ in $\mathcal{A}$ for $j \geq 1$.*

(ix) *For a fixed* $c_0 \in \widehat{\mathcal{A}}_0$*, a fixed* $(d_0, \ldots, d_r) \in \mathcal{N}$ *and for a fixed sequence of branches* $(k_1, j_1), \ldots, (k_r, j_r)$ *there are at most* $2r+1$ *different paths* $\mathbf{c} := c_0 \to_{(k_0,j_0)} \cdots \to_{(k_r,j_r)} c_r \in \mathcal{P}$ *such that* $\chi(\mathbf{c}) = (d_0, \ldots, d_r)$*. Further, for* $q \in \{0, 1, \ldots, r-1\}$ *and fixed* $d_0, d_1, \ldots, d_q \in \mathcal{A}_r \cap \mathcal{G}$ *there are at most* 4 *different* $a \in \mathcal{G}$ *such that there exists* $d_{q+2}, d_{q+3}, \ldots, d_r \in \mathcal{A} \cup \mathcal{G}$ *with* $(d_0, d_1, \ldots, d_q, a, d_{q+2}, d_{q+3}, \ldots, d_r) \in \chi(\mathcal{P})$*.*

In [23] a constructive proof is given. The same construction also works in our case. Note that in contrast with [23], our T is multi-valued, this is why we have to specify which branches we use in parts *(ii)* and *(ix)*. As this construction is very long, technical and presents nothing new we omit to repeat it here.

Instead, we give a general explanation on the statements to help building intuition on the structure of the Markocv diagrams. Using the lemma's notations, it states that for any big integer $r \in \mathbb{N}$ we can find a small number $\delta > 0$ such that the r-th level of the variants $\mathcal{A}_r$ and $\widehat{\mathcal{A}}_r$ have similar structures, given that $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are δ-close to each other.

Namely, $\widehat{A}_r$ contains a bijective copy of $\mathcal{A}_r$ which we denote by $\mathcal{B}_0$. According to (iv), there is a one-to-one correspondence between the edges of $(\mathcal{A}_r, \to)$ and $(\mathcal{B}_0, \to)$. As (v) and (vi) implies, the same relation is present between the intervals corresponding to the elements of $\mathcal{A}_r$ and $\mathcal{B}_0$. That is, a big initial part of the diagram of $\mathcal{F}$ is contained in the diagram of $\widehat{\mathcal{F}}$.

Assume that $\mathcal{F}$ has an inner critical point with a periodic orbit. Since perturbing the parameters of the original IFS may result in destroying this periodic orbit, we cannot expect $\widehat{\mathcal{F}}$ to share the same Markov diagram with $\mathcal{F}$, which is a representation of the orbit structure of these dynamical systems. In this case $\widehat{\mathcal{A}}_r \setminus \mathcal{B}_0$ is non-empty, this is the set we call $\mathcal{B}_1 \cup \mathcal{B}_2$.

Let $d \in \widehat{\mathcal{A}}_r \setminus \mathcal{B}_0$ be a code which appears in the variant $\widehat{\mathcal{A}}_r$ but isn't included in $\mathcal{A}_r$. It must code the successor of a point that is mapped onto an $a_l \in E_{\mathcal{Y}}$ by the expansive mapping T, for some $l \in [2N]$. Critical points have assigned codes, a_l is already coded by $(l, \emptyset)$, that is why d is not in $\mathcal{A}_r$.

If $a \in \mathcal{K}_{\mathcal{Y}}$, then $d \in \mathcal{B}_2$. By the definition of the orbit graph of critical points $(\mathcal{G}, \to)$, we can relate the elements of $\mathcal{B}_2$ to the nodes of $\mathcal{G}$. It also suggests that $(\mathcal{B}_2, \to)$ is a closed subgraph of $(\widehat{\mathcal{A}}_r, \to)$.

If $a \in E_{\mathcal{Y}} \setminus \mathcal{K}_{\mathcal{Y}}$, then we say $d \in \mathcal{B}_1$. These points move along with the endpoint of an element of the partition $\mathcal{Y}$. Since this endpoint is not a critical point, we can always find an element of $\mathcal{B}_0$ which is mapped onto the same monotonicity interval by $\widehat{A}$, according to the last two lines of (viii).

6.1.2 Continuity of the natural dimension

We prove here that the natural dimension of a limit-irreducible CPLIFS is always lower semi-continuous, and we give a condition for its upper semi-continuity as well. The proofs follow the lines of the proof of [23, Theorem 1] and [23, Theorem 2] respectively.

Theorem 6.1.7. *Let $\mathcal{F}$ be a limit-irreducible CPLIFS with attractor Λ, and let $s \in (0, \dim_{\mathrm{H}} \Lambda)$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all CPLIFS $\widehat{\mathcal{F}}$ which is δ-close to $\mathcal{F}$*

$$\Phi(s) - \varepsilon < \widehat{\Phi}(s),$$

where $\Phi(s)$ and $\widehat{\Phi}(s)$ stand for the natural pressure function of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively.

Proof. Let $\mathcal{Y}$ be the limit-irreducible partition of $\mathcal{F}$, and let $(\mathcal{D}, \rightarrow)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$.

Fix $\varepsilon > 0$ and write K for the maximum number of overlapping branches. By Proposition 4.2.1, there exists a $\mathcal{C} \subset \mathcal{D}$ finite subset such that

$$\varrho(\mathbf{F}(s)) - \frac{\varepsilon}{2} \leq \varrho(\mathbf{F}_{\mathcal{C}}(s)), \tag{6.5}$$

where $\mathbf{F}(s)$ is the matrix associated to the Markov diagram. As $\mathcal{C}$ is finite it must be contained in $\mathcal{D}_r$ for some $r \geq 1$. By Lemma 6.1.6, there exists a $\delta \in (0, \frac{\varepsilon}{2sK})$ such that the conclusions of Lemma 6.1.6 are true with respect to r if $\widehat{\mathcal{F}}$ is δ-close to $\mathcal{F}$.

Let $\widehat{\mathcal{Z}}_0$ be the monotonicity partition of $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{Y}}$ be a finite refinement of $\widehat{\mathcal{Z}}_0$ which is δ-close to $\mathcal{Y}$. Let $(\mathcal{A}, \rightarrow), (\widehat{\mathcal{A}}, \rightarrow)$ be the variants of the Markov diagram of $\mathcal{F}, \widehat{\mathcal{F}}$ with repsect to $\mathcal{Y}$ and $\widehat{\mathcal{Y}}$ respectively, obtained by applying Lemma 6.1.6.

By (6.5) and Remark 6.1.5, there exists a $\mathcal{C} \subset \mathcal{A}_r$ such that

$$\Phi(s) - \varepsilon = \log \varrho(\mathbf{F}(s)) - \varepsilon \leq \log \varrho(\mathbf{F}^{\mathcal{A}}_{\mathcal{C}}(s)) - \frac{\varepsilon}{2}, \tag{6.6}$$

where the first equality follows from Corollary 4.2.4.

We write ρ and $\widehat{\rho}$ for the slopes of the functions of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively. Consider the matrix $\mathbf{F}^{\widehat{\mathcal{A}}}_{\phi(\mathcal{C})}(s)$, where $\phi : \mathcal{A}_r \rightarrow \widehat{\mathcal{A}}_r$ is the mapping described in Lemma 6.1.6. As $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are δ-close, for any $k \in [m]$ and $j \in [l(k)]$ we have

$$\log \rho_{k,j} - \delta \leq \log \widehat{\rho}_{k,j}.$$

Using that every entry in $\mathbf{F}^{\mathcal{A}}_{\mathcal{C}}(s)$ is a sum of at most K many elements of the form $\rho^s_{k,j}$, by parts $(iv), (v)$ and (vi) of Lemma 6.1.6 we have the following relation between the elements of $\widehat{\mathbf{F}}^{\widehat{\mathcal{A}}}_{\phi(\mathcal{C})}(s)$ and $\mathbf{F}^{\mathcal{A}}_{\mathcal{C}}(s)$.

$$\forall c, d \in \mathcal{C} : \quad \left[\widehat{\mathbf{F}}^{\widehat{\mathcal{A}}}_{\phi(\mathcal{C})}(s)\right]_{\phi(c),\phi(d)} \geq \exp^{-\frac{\varepsilon}{2}} \left[\mathbf{F}^{\mathcal{A}}_{\mathcal{C}}(s)\right]_{c,d}, \tag{6.7}$$

since δ is smaller than $\frac{\varepsilon}{2sK}$. By (6.6), (6.7) and Lemma 4.1.8, we conclude the proof

$$\Phi(s) - \varepsilon \leq \log \varrho(\mathbf{F}^{\mathcal{A}}_{\mathcal{C}}(s)) - \frac{\varepsilon}{2} \leq \log \varrho(\widehat{\mathbf{F}}^{\widehat{\mathcal{A}}}_{\phi(\mathcal{C})}(s)) = \widehat{\Phi}_{\phi(\mathcal{C})}(s) \leq \widehat{\Phi}(s).$$

□

Theorem 6.1.8. *Let $\mathcal{F}$ be a limit-irreducible CPLIFS with attractor Λ, and let $\mathbf{G}(s)$ be the matrix defined in* (6.1)*. Fix an arbitrary $s \in (0, \dim_{\mathrm{H}} \Lambda)$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all CPLIFS $\widehat{\mathcal{F}}$ which are δ-close to $\mathcal{F}$*

$$\widehat{\Phi}(s) < \max\{\Phi(s), \log \varrho(\mathbf{G}(s))\} + \varepsilon,$$

where $\Phi(s)$ and $\widehat{\Phi}(s)$ stand for the natural pressure function of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively.

The proof is analogous to the proof of [23, Theorem 2].

Proof. Let $\mathcal{Y}$ be the limit-irreducible partition of $\mathcal{F}$, and let $(\mathcal{D}, \to)$ be the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$.

Fix $\varepsilon > 0$ and write K for the maximum number of overlapping branches. We define the value

$$R_0 := \exp\left(\max\left\{\Phi(s), \log \varrho(\mathbf{G}(s))\right\} + \varepsilon\right).$$

As $R_0 > e^{\frac{\varepsilon}{2}} \max\left\{e^{\Phi(s)}, \varrho(\mathbf{G}(s))\right\}$, we can choose an

$$R \in \left(e^{\frac{\varepsilon}{2}} \max\left\{e^{\Phi(s)}, \varrho(\mathbf{G}(s))\right\}, R_0\right).$$

It follows that $e^{-\frac{\varepsilon}{2}} R > \varrho(\mathbf{G}(s))$ and $e^{-\frac{\varepsilon}{2}} R > e^{\Phi(s)}$. By Lemma 6.1.4 and Corollary 4.2.4, it also follows that for any $(\mathcal{A}, \to)$ variant of the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$, we have $e^{-\frac{\varepsilon}{2}} R > \varrho(\mathbf{F}^{\mathcal{A}}(s))$.

By Gelfand's formula, for any bounded linear operator $\mathbf{M}$ the spectral radius satisfies $\varrho(\mathbf{M}) = \inf_{n \geq 1} \|\mathbf{M}^n\|^{\frac{1}{n}}$. Thus, there exists a $C \in \mathbb{R}$ such that

$$\sup_{n \in \mathbb{N}} e^{\frac{\varepsilon}{2} n} R^{-n} \|\mathbf{G}(s)^n\| \leq C, \tag{6.8}$$

and for every $(\mathcal{A}, \to)$ variant of the Markov diagram of $\mathcal{F}$ with respect to $\mathcal{Y}$

$$\sup_{n \in \mathbb{N}} e^{\frac{\varepsilon}{2} n} R^{-n} \|\mathbf{F}^{\mathcal{A}}(s)^n\| \leq C. \tag{6.9}$$

We may assume that

$$C \geq \max\left\{2, 8K^{r-1} e^{\frac{\varepsilon}{2}} R^{-1} \rho^s_{\max}\right\},$$

where $\rho_{\max} = \max_{k\in[m],j\in[l(k)]} |\rho_{k,j}|$ is the biggest slope of the functions in $\mathcal{F}$ in absolute value. Fix an $r \in \mathbb{N}$ with

$$\sqrt[r]{(r+1)^2 C^3} R < R_0.$$

By Lemma 6.1.6, there exists a $\delta \in (0, \frac{\varepsilon}{2})$ such that the conclusions of Lemma 6.1.6 are true with respect to r if $\widehat{\mathcal{F}}$ is δ-close to $\mathcal{F}$.

Let $\widehat{\mathcal{Z}}_0$ be the monotonicity partition of $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{Y}}$ be a finite refinement of $\widehat{\mathcal{Z}}_0$ which is δ-close to $\mathcal{Y}$. Let $(\mathcal{A}, \to), (\widehat{\mathcal{A}}, \to)$ be the variants of the Markov diagram of $\mathcal{F}, \widehat{\mathcal{F}}$ with repsect to $\mathcal{Y}$ and $\widehat{\mathcal{Y}}$ respectively, obtained by applying Lemma 6.1.6.

Like in the proof of Theorem 6.1.7, we write ρ and $\widehat{\rho}$ for the slope of the functions of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively. Let $\phi : \mathcal{A}_r \to \widehat{\mathcal{A}}_r$ be the bijective mapping described in Lemma 6.1.6. As $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are δ-close, for any $k \in [m]$ and $j \in [l(k)]$ we have

$$\log \widehat{\rho}_{k,j} \le \log \rho_{k,j} + \delta \le \log \rho_{k,j} + \frac{\varepsilon}{2}. \tag{6.10}$$

Let $\mathcal{P}_c$ be the set of all paths $c_0 \to_{(k_0,j_0)} c_1 \to_{(k_1,j_1)} \cdots \to_{(k_{r-1},j_{r-1})} c_r$ of length r in $\widehat{\mathcal{A}}_r$ with $c_0 = c$. To prove the statement of the theorem, it is enough to give an upper bound on the sum of the weights of paths of length r in $\widehat{\mathcal{A}}_r$ starting at the same node, since

$$\begin{aligned} \Big(\varrho(\widehat{\mathbf{F}}(s))\Big)^r &\le \|\widehat{\mathbf{F}}(s)^r\|_\infty = \|\widehat{\mathbf{F}}^{\widehat{\mathcal{A}}_r}(s)^r\|_\infty \qquad (6.11)\\ &= \sup_{c\in\widehat{\mathcal{A}}_0} \sum_{\mathcal{P}_c} \prod_{n=0}^{r-1} |\widehat{\rho}_{k_n,j_n}|^s, \end{aligned}$$

where the sum is taken over all paths $c_0 \to_{(k_0,j_0)} c_1 \to_{(k_1,j_1)} \cdots \to_{(k_{r-1},j_{r-1})} c_r$ in $\mathcal{P}_c$.

Fix $c \in \widehat{\mathcal{A}}_0$. Then by (i) and (iii) of Lemma 6.1.6, there exists a unique $d \in \mathcal{A}_0$ with $\phi(d) = c$. Using the notation of Lemma 6.1.6, we write $\widehat{\mathcal{A}}_r = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ such that $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_2$ are closed in $\widehat{\mathcal{A}}_r$. Then, we can partition the paths in $\mathcal{P}_c$ based on the index of the last node in $\mathcal{B}_0 \cup \mathcal{B}_1$. For $q \in \{0, 1 \ldots, r\}$, let $\mathcal{P}_c(q)$ be the set of all paths $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}_c$ with $q = \max\{j \in \{0, 1 \ldots, r\} : c_j \in \mathcal{B}_0 \cup \mathcal{B}_1\}$. Thus $\mathcal{P}_c = \cup_{s=0}^r \mathcal{P}_c(q)$. Define for $q \in \{0, 1 \ldots, r\}$

$$H_c(q) := \sum_{\mathcal{P}_c(q)} \prod_{n=0}^{r-1} |\widehat{\rho}_{k_n,j_n}|^s.$$

Writing $H_c := \sum_{q=0}^{r} H_c(q)$, we can reformulate (6.11) as

$$\left(\varrho(\widehat{\mathbf{F}}(s))\right)^r \le \sup_{c \in \widehat{\mathcal{A}}_0} H_c. \tag{6.12}$$

Let $q \in \{0, 1, \dots, r\}$. If $\mathbf{c} := c_0 \to_{(k_0,j_0)} c_1 \to_{(k_1,j_1)} \cdots \to_{(k_{r-1},j_{r-1})} c_r \in \mathcal{P}_(c)$ and $(d_0, d_1, \dots, d_r) = \chi(\mathbf{c})$, then (i) and (viii) of Lemma 6.1.6 give $d_j \in \mathcal{A}_r$ for all $j \le q$, $d_0 = d$ and $d_j \in \mathcal{G}$ for all $j > q$. Hence if $q \ge 1$, then $d_0 \to_{(k_0,j_0)} d_1 \to_{(k_1,j_1)} \cdots \to_{(k_{q-1},j_{q-1})} d_q$ is a path of length q in $\mathcal{A}_r$ with $d_0 = d$. Moreover, (viii) of Lemma 6.1.6 also guarantees that $\widehat{A}(c_j)$ is $\widehat{\mathfrak{Y}}$-close to $\widehat{A}(\phi(d_j))$ for all $j \in \{0, 1, \dots, q\}$. Thus by (6.10)

$$\prod_{n=0}^{q-1} |\widehat{\rho}_{k_n,j_n}|^s \le e^{\frac{\varepsilon}{2} sq} \prod_{n=0}^{q-1} |\rho_{k_n,j_n}|^s, \quad \text{if } q \ge 1.$$

If $q \le r-2$, then by (iv), (vi) and (viii) of Lemma 6.1.6, the path $d_{q+1} \to_{(k_{q+1},j_{q+1})} \cdots \to_{(k_{r-1},j_{r-1})} d_r$ is of length $r-q-1$ in $\mathcal{G}$, and $\widehat{A}(c_j)$ is $\widehat{\mathfrak{Y}}$-close to $\widehat{A}(\phi(\tilde{d}_j))$ for all $j \in \{q+1, q+2, \dots, r\}$, where $\tilde{d}_j \in \mathcal{A}_0$ satisfies $d_j \in A(\tilde{d}_j)$. Hence (6.10) gives

$$\prod_{n=q+1}^{r-1} |\widehat{\rho}_{k_n,j_n}|^s \le e^{\frac{\varepsilon}{2} s(r-q-1)} \prod_{n=q+1}^{r-1} |\rho_{k_n,j_n}|^s, \quad \text{if } q \le r-2.$$

We apply (ix) of Lemma 6.1.6 to obtain for $1 \le q \le r-2$

$$\begin{aligned}
H_c(q) &\le K^{r-1}(2r+1)e^{\frac{\varepsilon}{2}sr} \sum_{(d_0,d_1,\dots,d_r)\in\chi(\mathcal{P}_c(q))} |\rho_{k_q,j_q}|^s \prod_{n=0}^{q-1} |\rho_{k_n,j_n}|^s \prod_{n=q+1}^{r-1} |\rho_{k_n,j_n}|^s \\
&\le K^{r-1} 8 e^{\frac{\varepsilon}{2}sr} \rho_{\max}^s (r+1) \left(\sum_{d=d_0 \to_{(k_0,j_0)} \cdots \to_{(k_{q-1} j_{s-1})} d_q} \prod_{n=0}^{q-1} |\rho_{k_n,j_n}|^s \right) \\
&\cdot \left(\sup_{a \in \mathcal{G}} \sum_{a_0 = a \to_{(\tilde{k}_0,\tilde{j}_0)} \cdots \to_{(\tilde{k}_{r-q-2},\tilde{j}_{r-q-2})} a_{r-q-1}} \prod_{n=0}^{r-q-2} |\rho_{\tilde{k}_n,\tilde{j}_n}|^s \right).
\end{aligned}$$

Keep in mind that according to (ix) of Lemma 6.1.6, at most $2r+1$ paths has the same image by χ, but only if they use the same edges. Due to the possible overlappings in $\mathcal{F}$ and $\widehat{\mathcal{F}}$, there might be parallel edges in the Markov diagrams, so it is possible that we can walk the same path using completely different edges. That is why we need the K^{r-1} multiplier.

Recall that $C \geq 8K^{r-1}e^{\frac{\varepsilon}{2}sr}R^{-1}\rho_{\max}^s$. By (6.8) and (6.9) we obtain

$$H_c(q) \leq CR(r+1)CR^qCR^{r-q-1} = (r+1)C^3R^r.$$

Similarly, the same bound holds for the remaining three values of q

$$\begin{aligned} H_c(0) &\leq K^{r-1}8e^{\frac{\varepsilon}{2}sr}\rho_{\max}^s(r+1)\left(\sup_{a\in\mathcal{G}} \sum_{a_0=a\to_{(\tilde{k}_0,\tilde{j}_0)}\cdots\to_{(\tilde{k}_{r-2},\tilde{j}_{r-2})}a_{r-1}} \prod_{n=0}^{r-2}|\rho_{\tilde{k}_n,\tilde{j}_n}|^s\right) \\ &\leq CR(r+1)CR^{r-1} \leq (r+1)C^3R^r, \\ H_c(r-1) &\leq K^{r-1}8e^{\frac{\varepsilon}{2}sr}\rho_{\max}^s(r+1)\left(\sum_{d=d_0\to_{(k_0,j_0)}\cdots\to_{(k_{r-2}j_{r-2})}d_{r-1}} \prod_{n=0}^{r-2}|\rho_{k_n,j_n}|^s\right) \\ &\leq CR(r+1)CR^{r-1} \leq (r+1)C^3R^r, \\ H_c(r) &\leq K^{r-1}2e^{\frac{\varepsilon}{2}sr}(r+1)\left(\sum_{d=d_0\to_{(k_0,j_0)}\cdots\to_{(k_{r-1}j_{r-1})}d_r} \prod_{n=0}^{r-1}|\rho_{k_n,j_n}|^s\right) \\ &\leq C(r+1)CR^r \leq (r+1)C^3R^r. \end{aligned}$$

Thus, $H_c = \sum_{q=0}^r H_c(q) \leq (r+1)^2C^3R^r$ for all $c \in \widehat{\mathcal{A}}_0$. Then (6.12) gives

$$\varrho(\widehat{\mathbf{F}}(s)) \leq \sqrt[r]{(r+1)^2C^3} \cdot R < R_0. \tag{6.13}$$

We conclude the proof by combining (6.13) with Lemma 4.1.8

$$\widehat{\Phi}(s) \leq \log\varrho(\widehat{\mathbf{F}}(s)) < \log R_0 = \max\{\Phi(s), \log\varrho(\mathbf{G}(s))\} + \varepsilon.$$

□

We have seen in Lemma 4.3.8 that a CPLIFS with no exact overlapping is limit-irreducible. Therefore, as a consequence of Theorem 6.1.7, Theorem 6.1.8 and Lemma 6.1.1, Theorem 6.0.3 follows.

Remark 6.1.9. *By the proof of Lemma 6.1.1, it also follows that the natural dimension of a CPLIFS changes continuously with respect to its parameters if the critical points, which are breaking point images, does not have periodic orbits.*

That is the natural dimension of a CPLIFS $\mathcal{F}$ changes continuously with respect to the parameters that define $\mathcal{F}$ (breaking points, slopes of functions, translation parameters) if there are no exact overlappings. This result combined with Theorem 4.2.2 yields an analogous result on the Hausdorff dimension of the attractor.

Remark 6.1.10. *Let $\mathcal{F}$ be a CPLIFS whose generated self-similar IFS satisfies the ESC. Fix an arbitrary $\varepsilon > 0$, and let $\mathfrak{P}$ be the property that the CPLIFS $\mathcal{F}'$ satisfies*

$$|\dim_{\mathrm{H}} \Lambda - \dim_{\mathrm{H}} \widehat{\Lambda}| < \varepsilon,$$

where Λ and $\widehat{\Lambda}$ are the attractors of $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively.

Then there exists a $\delta > 0$ such that $\mathfrak{P}$ is a $\dim_{\mathrm{P}}$*-typical property of CPLIFSs that are δ-close to $\mathcal{F}$.*

6.2 Lebesgue measure of the attractor

Theorem 6.2.1. *Fix a type $\boldsymbol{\ell}$ and a vector of slopes $\boldsymbol{\rho} \in \mathfrak{R}^{\boldsymbol{\ell}}$. If all elements of $\boldsymbol{\rho}$ are positive, then for $\mathcal{L}^{L+m}$-almost every $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$*

$$s_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} > 1 \implies \mathcal{L}\left(\Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}\right) > 0, \tag{6.14}$$

where $s_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$ and $\Lambda^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$ are the natural dimension and attractor of the continuous piecewise linear iterated function system defined by the parameters $(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{\rho})$.

Proof. Let $E_{\mathrm{ESC}} \subset \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ be the set of parameters for which the generated self-similar IFS $\mathcal{S}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}$ satisfies the ESC. Hochman proved [9, Theorem 1.10] that

$$\mathcal{L}^{L+m}(\mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \setminus E_{\mathrm{ESC}}) = 0.$$

Thus it is enough to focus on the elements of E_{ESC}. Fix an arbitrary $(\mathfrak{b}, \boldsymbol{\tau}) \in E_{\mathrm{ESC}}$ and assume that $s_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} > 1$. Set $\varepsilon := \Phi_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}(1)$. The natural pressure function is strictly decreasing, and $s_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} > 1$, hence $\varepsilon > 0$.

Since our parameter space $\mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m \subset \mathbb{R}^{m+L}$ is σ-compact, it suffices to show that (6.14) holds for $\mathcal{L}^{L+m}$-almost every element of a well defined open neighbourhood around any point $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$. As a CPLIFS whose generated self-similar IFS satisfies the ESC is always limit-irreducible, according to Theorem 6.1.7, there exists a $B((\mathfrak{b}, \boldsymbol{\tau}), \delta) \subset \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ open ball around $(\mathfrak{b}, \boldsymbol{\tau})$ of radius δ such that for all $(\widehat{\mathfrak{b}}, \widehat{\boldsymbol{\tau}}) \in B((\mathfrak{b}, \boldsymbol{\tau}), \delta)$

$$\forall s \in [0, s_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}) : \quad \Phi_{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})}(s) - \varepsilon < \Phi_{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}},\boldsymbol{\rho})}(s).$$

It follows that

$$\forall (\widehat{\mathfrak{b}}, \widehat{\boldsymbol{\tau}}) \in B((\mathfrak{b}, \boldsymbol{\tau}), \delta) : \quad s_{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}},\boldsymbol{\rho})} > 1.$$

Write $\mathbf{t}_{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}})}$ for the vector of translations of $\mathcal{S}^{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}},\boldsymbol{\rho})}$. By [15, Theorem 1], for $\mathcal{L}^{L+m}$-almost every $\mathbf{t}_{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}})} \in \mathbb{R}^{L+m}$ the assertion of the theorem holds for

$\mathcal{S}^{(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}},\boldsymbol{\rho})}$. Then by Claim 3.1.1, $\mathcal{L}^{L+m}$-almost every $(\widehat{\mathfrak{b}},\widehat{\boldsymbol{\tau}}) \in B((\mathfrak{b},\boldsymbol{\tau}),\delta)$ satisfies (6.14). As $(\mathfrak{b},\boldsymbol{\tau})$ was arbitrary, the assertion of the theorem follows.

□

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