Fractal dimensions of piecewise linear iterated function systems

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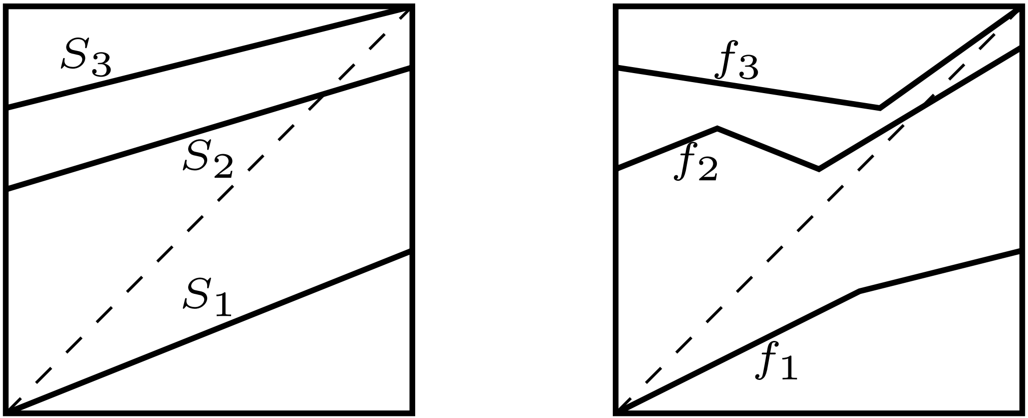
DAGGER seminar, 5 June 2023



Self-similar IFS
$\mathcal{S} = \{S_1, S_2, S_3\}$

CPLIFS
$\mathcal{F} = \{f_1, f_2, f_3\}$

The Main Result

Theorem (Raith, Simon, P.)

For packing dimension typical CPLIFS $\mathcal{F}$

(1) $$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}.$$

The meaning of "packing dimension typical": the packing dimension of the set of parameters of the exceptional CPLIFSs is less than the dimension of the parameter space.

An Iterated Function System (IFS) $\mathcal{F} = \{f_k\}_{k=1}^m$ on the line is a finite list of strict contractions on $\mathbb{R}$.

The attractor of the IFS $\mathcal{F}$ is the unique non-empty compact set that satisfies

(2) $$\Lambda = \bigcup_{k=1}^{m} f_k(\Lambda).$$

By iterating formula (2), one obtains

(3) $$\Lambda = \bigcup_{(i_1,\dots,i_n)\in[m]^n} f_{i_1\dots i_n}(\Lambda).$$

Here we used the common notation $f_{i_1\dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$.

Let I be the smallest non-empty compact interval such that $f_i(I) \subset I$ for all $i \in [m] := \{1, \ldots, m\}$.

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in[m]^n} I_{i_1\ldots i_n}, \tag{4}$$

where $I_{i_1\ldots i_n} := f_{i_1\ldots i_n}(I)$ are the cylinder intervals.
Thus these intervals form a natural cover of the attractor.

The natural dimension

(5) $$\Phi(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s.$$

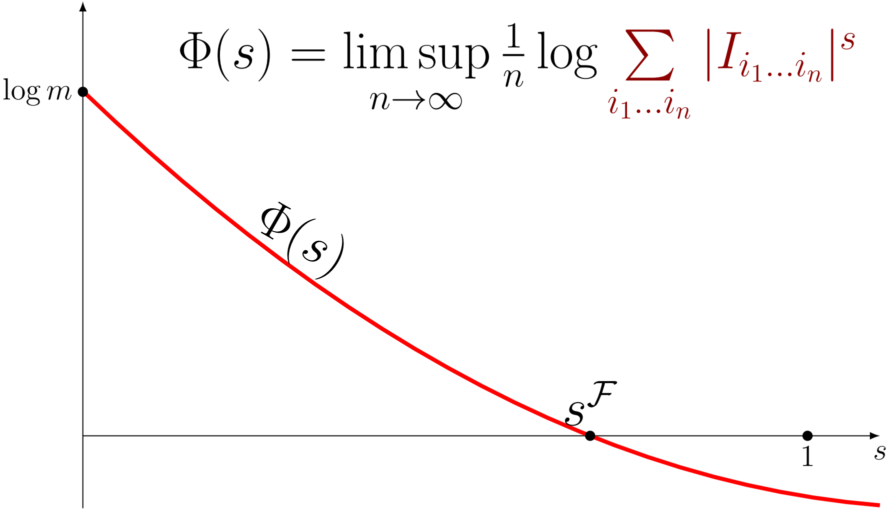
We obtain $\Phi(s)$ as a special case of the non-additive upper capacity topological pressure introduced by Barreira[1]. The unique zero of this function is the natural dimension of $\mathcal{F}$.

(6) $$s^{\mathcal{F}} := (\Phi)^{-1}(0).$$

[1] Luis M Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems.
Ergodic Theory and Dynamical Systems, 16(5):871–928, 1996

The natural dimension

$$\Phi(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s$$



The Hausdorff dimension

The t-dimensional Hausdorff measure of the attractor is

$$(7) \quad \mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \Lambda \subset \bigcup_{i=1}^{\infty} A_i, \ |A_i| \leqslant \delta \right\} \right\},$$

where the infimum is taken over all $\{A_i\}$ covers.
The Hausdorff dimension of Λ is defined as

$$(8) \quad \dim_{\mathrm{H}} \Lambda = \inf\{t : \mathcal{H}^t(\Lambda) = 0\} = \sup\{t : \mathcal{H}^t(\Lambda) = \infty\}.$$

The packing dimension

The t-dimensional packing measure of the attractor is

(9) $$\tilde{\mathcal{P}}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \sup \left\{ \sum_{i=1}^{\infty} |B_i|^t : \{\bar{B}_i\} \text{ is a } \delta\text{-packing of } \Lambda \right\} \right\},$$

(10) $$\mathcal{P}^t(\Lambda) = \inf \left\{ \sum_{i=1}^{\infty} \tilde{\mathcal{P}}^t(E_i) : \Lambda \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

The packing dimension of Λ is defined as

(11) $$\dim_{\mathrm{P}} \Lambda = \inf\{t : \mathcal{P}^t(\Lambda) = 0\} = \sup\{t : \mathcal{P}^t(\Lambda) = \infty\}.$$

Barreira[2] also showed that

$$\dim_{\mathrm{H}} \Lambda \leqslant \overline{\dim}_{\mathrm{B}} \Lambda \leqslant \min\left\{1, s^{\mathcal{F}}\right\}. \tag{12}$$

Under what condition do we have equality?

It is easy to see that $\dim_{\mathrm{H}} \Lambda < s^{\mathcal{F}}$ if some cylinder intervals are identical. Thus we need to require some separation for the system.

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[2] Luis M Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergodic Theory and Dynamical Systems*, 16(5):871–928, 1996

Separation Conditions

Consider the IFS $\mathcal{F} = \{f_k\}_{k=1}^m$.
We say that $\mathcal{F}$ satisfies the Strong Separation Property (SSP) if

$$\forall i, j \in [m], i \neq j : f_i(\Lambda) \cap f_j(\Lambda) = \emptyset.$$

We say that $\mathcal{F}$ satisfies the Open Set Condition (OSC) if $\exists U$ open set such that $\forall i \in [m] : f_i(U) \subset U$ and

$$\forall i, j \in [m], i \neq j : f_i(U) \cap f_j(U) = \emptyset.$$

Both of these conditions guarantee that $\dim_{\mathrm{H}} \Lambda = s^{\mathcal{F}}$.

Self-similar IFS

If our iterated function system is of the form

$$\mathcal{F} = \{f_k(x) = r_k \cdot x + t_k\}_{k=1}^{m}$$

then $\mathcal{F}$ is called self-similar. In this case

$$\Phi(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \ldots i_n} |r_{i_1} \cdots r_{i_n}|^s = \log \sum_{i=1}^{m} |r_i|^s \tag{13}$$

$$\Phi(s^{\mathcal{F}}) = 0 \iff \sum_{k=1}^{m} |r_k|^{s^{\mathcal{F}}} = 1. \tag{14}$$

Hence $s^{\mathcal{F}}$ is the similarity dimension.

Exponential Separation Condition

The distance of two similarity mappings $g_1(x) = r_1 x + \tau_1$ and $g_2(x) = r_2 x + \tau_2$, $r_1, r_2 \in (-1, 1) \setminus \{0\}$, on $\mathbb{R}$.

$$\operatorname{dist}(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } r_1 = r_2; \\ \infty, & \text{otherwise.} \end{cases} \tag{15}$$

We say that the self-similar IFS $\mathcal{F}$ satisfies the Exponential Separation Condition (ESC) if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^{\infty}$ such that

$$\operatorname{dist}(f_{\bar{\imath}}, f_{\bar{\jmath}}) \geqslant c^{n_\ell} \text{ for all } \ell \text{ and for all } \bar{\imath}, \bar{\jmath} \in \{1, \dots, m\}^{n_\ell}, \ \bar{\imath} \neq \bar{\jmath}.$$

Self-similar IFS 2

Hochman [3] proved that for any self-similar IFS on the line that satisfies the ESC we have

$$\dim_{\mathrm{H}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}.$$

We managed to extend this result to CPLIFS, with the help of Markov diagrams.

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[3] Michael Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Annals of Mathematics*, pages 773–822, 2014

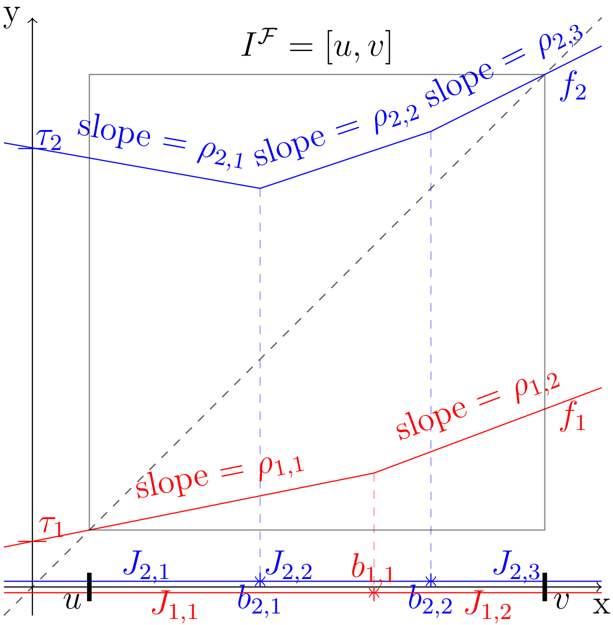
Introduction

Markov Diagrams

Proof of the Main Theorem

Proof of Theorem 3.4

Limit-irreducibility



$\mathcal{F} = \{f_k\}_{k=1}^m$, $\tau_k := f_k(0)$,

f_k has $l(k)$ breaking points $\{b_{k,1}, \ldots, b_{k,l(k)}\}$.

The type of $\mathcal{F}$ is $\boldsymbol{\ell} = (l(1), \ldots, l(m))$

$L := l(1) + \cdots + l(m)$.

Let $I_k := f_k(I)$ and $\mathcal{I} = \cup_{k=1}^m I_k$. We define the expanding multi-valued mapping associated to $\mathcal{F}$ as

(16) $$T : \mathcal{I} \mapsto \mathcal{P}(\mathcal{P}(I))$$

(17) $$T(y) := \{\{x \in I : f_k(x) = y\}\}_{k=1}^m.$$

For $k \in [m], j \in [l(k)+1]$, we define $f_{k,j} : J_{k,j} \mapsto I_k$ as the uniqe linear function that satisfies $f_k(x) = f_{k,j}(x), \forall x \in J_{k,j}$.

We refer to the linear functions

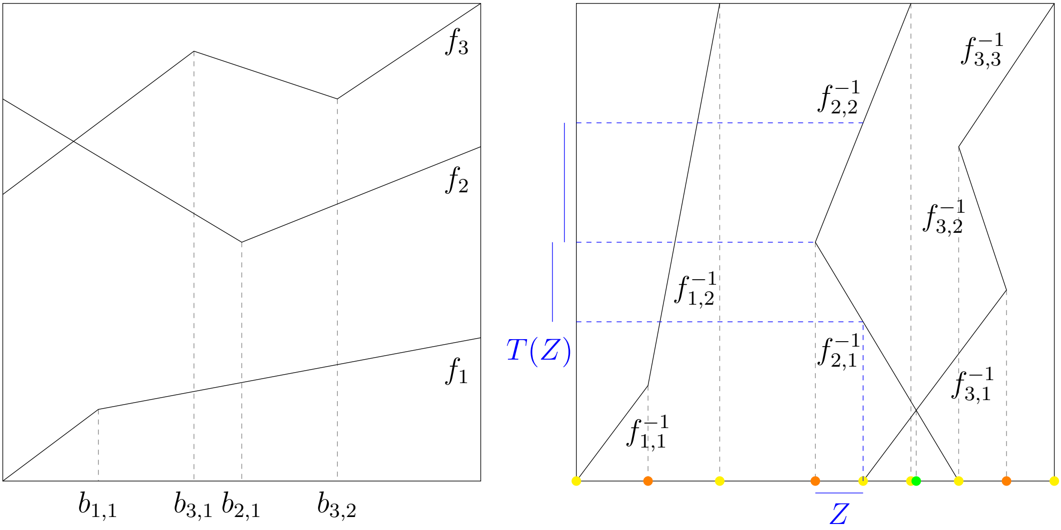
$$\forall k \in [m], \forall j \in [l(k)+1] : f_{k,j}^{-1}$$

as the branches of T.

We define the set of critical points as

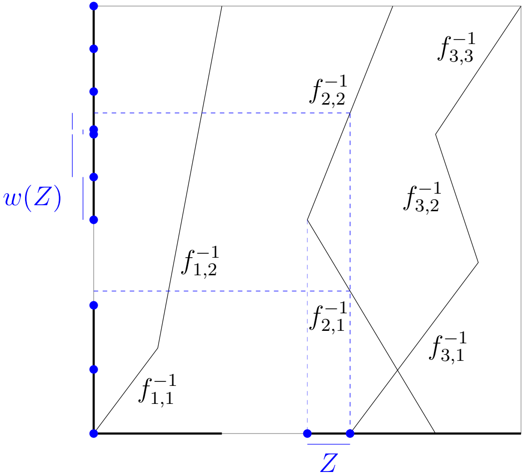
$$\mathcal{K} := \cup_{k=1}^{m}\{f_k(0), f_k(1)\} \bigcup \cup_{k=1}^{m} \cup_{j=1}^{l(k)} f_k(b_{k,j}) \bigcup$$
$$\left\{x \in \mathcal{I} \middle| \exists k_1, k_2 \in [m], \exists j_1 \in [l(k_1)], \exists j_2 \in [l(k_2)] : f_{k_1,j_1}^{-1}(x) = f_{k_2,j_2}^{-1}(x)\right\}$$

The associated multi-valued mapping



We call the partition of $\mathcal{I}$ into closed intervals defined by the set of critical points $\mathcal{K}$ the monotonicity partition $\mathcal{Z}_0$ of $\mathcal{F}$. We call its elements monotonicity intervals.

That is, above monotonicity intervals T is always linear, and branches can only take the same value at the boundary.



Let $Z \in \mathcal{Z}_0$. We write $Z \to D$ for the successors of Z.

$$\exists Z_0 \in \mathcal{Z}_0, Z' \in T(Z) :$$

$$D = Z_0 \cap Z'$$

Further, we write $Z \to_{k,j} D$ if

$$\exists Z_0 \in \mathcal{Z}_0 : D = Z_0 \cap f_{k,j}^{-1}(Z).$$

The set of successors of Z is $w(Z) := \{D | Z \to D\}$.

Following Hofbauer and Raith, we say that $(\mathcal{D}, \rightarrow)$ is the Markov Diagram of $\mathcal{F}$ with respect to $\mathcal{Z}_0$ if $\mathcal{D}$ is the smallest set containing $\mathcal{Z}_0$ such that $\mathcal{D} = w(\mathcal{D})$.

We can similarly define the Markov diagram of $\mathcal{F}$ with respect to any finite partition $\mathcal{Z}_0'$ of $\mathcal{I}$.

One can imagine the Markov diagram as a (potentially infinitely big) directed graph, with vertex set $\mathcal{D}$.

Between $C, D \in \mathcal{D}$, we have a directed edge $C \to D$ if and only if $D \in w(C)$. We call the Markov diagram irreducible if there exists a directed path between any two intervals $C, D \in \mathcal{D}$.

Since the functions of a CPLIFS are always continuous on $\mathbb{R}$, we can always assume that $(\mathcal{D}, \to)$ is irreducible.

Associated matrix

We define the matrix $\mathbf{F}(s) := \mathbf{F}_{\mathcal{D}}(s)$ indexed by the elements of $\mathcal{D}$ as

$$[\mathbf{F}(s)]_{C,D} := \begin{cases} \sum_{(k,j):C \to_{(k,j)} D} \left| f'_{k,j} \right|^s, \text{ if } C \to D \\ 0, \text{ otherwise.} \end{cases} \tag{18}$$

This matrix is often associated to self-similar graph directed iterated function systems. When the diagram is finite, our system is actually a self-similar GDIFS.

Let $\mathcal{C} \subset \mathcal{D}$. We write $\mathcal{E}_{\mathcal{C}}(n)$ for the set of n-length directed paths in the subgraph $(\mathcal{C}, \rightarrow)$.

Assume that $(\mathcal{C}, \rightarrow)$ is irreducible. Each path in $(\mathcal{C}, \rightarrow)$ of infinite length represents a point in the invariant set $\Lambda_{\mathcal{C}} \subset \Lambda$. We define the natural pressure of these sets as

$$\Phi_{\mathcal{C}}(s) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\mathbf{k}} |I_{\mathbf{k}}|^s, \tag{19}$$

where the sum is taken over all $\mathbf{k} = (k_1, \ldots k_n)$ for which $\exists j_1, \ldots j_n : ((k_1, j_1), \ldots, (k_n, j_n)) \in \mathcal{E}_{\mathcal{C}}(n)$.

As an operator, $(\mathbf{F}_{\mathcal{D}}(s))^n$ is always bounded in the l^∞-norm. Thus we can define

$$\varrho(\mathbf{F}_{\mathcal{C}}(s)) := \lim_{n\to\infty} \|(\mathbf{F}_{\mathcal{C}}(s))^n\|_\infty^{1/n}.$$

Lemma 2.1

Let $\mathcal{C} \subset \mathcal{D}$. If $(\mathcal{C}, \to)$ is irreducible, then

$$\Phi_{\mathcal{C}}(s) \leqslant \log \varrho(\mathbf{F}_{\mathcal{C}}(s)). \tag{20}$$

If $(\mathcal{C}, \to)$ is irreducible and finite, then

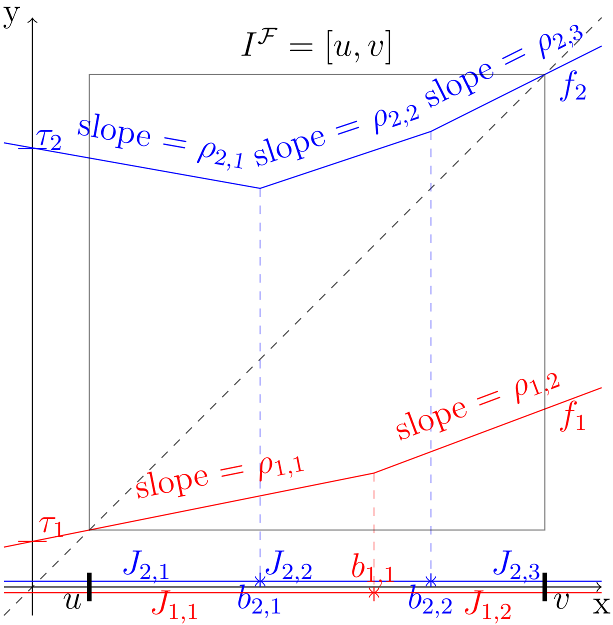
$$\Phi_{\mathcal{C}}(s) = \log \varrho(\mathbf{F}_{\mathcal{C}}(s)). \tag{21}$$

The Main Result

Theorem (Raith, Simon, P.)

For packing dimension typical CPLIFS $\mathcal{F}$

(22) $$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}.$$



$\mathcal{F} = \{f_k\}_{k=1}^m$, $\tau_k := f_k(0)$,

f_k has $l(k)$ breaking points $\{b_{k,1}, \ldots, b_{k,l(k)}\}$.

The type of $\mathcal{F}$ is $\boldsymbol{\ell} = (l(1), \ldots, l(m))$

$L := l(1) + \cdots + l(m)$.

Packing dimension typicality

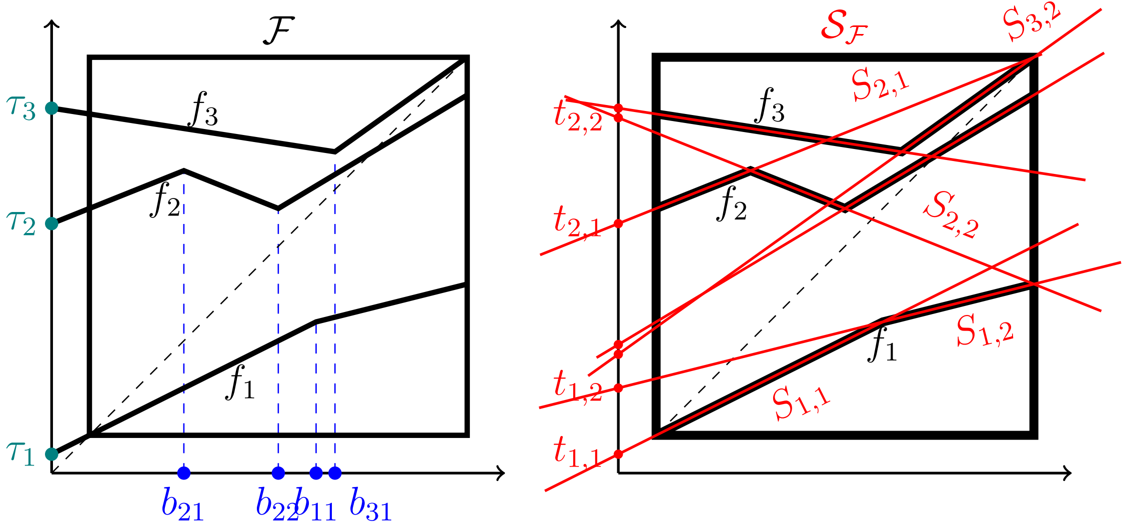
Fix a type $\boldsymbol{\ell} = (l(1), \ldots, l(m))$ and a vector of contractions $\boldsymbol{\rho} \in ((-1,1)\backslash\{0\})^{L+m}$. Let $\mathfrak{P}$ be a property that makes sense for every CPLIFS, and consider the exceptional set

(23) $$E_{\boldsymbol{\ell}}^{\boldsymbol{\rho}} =: \left\{(\mathfrak{b}, \boldsymbol{\tau}) \in \mathbb{R}^{L+m} : \mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \text{ does not have property } \mathfrak{P}\right\}.$$

We say that property $\mathfrak{P}$ holds $\dim_{\mathrm{P}}$-typically if for all type $\boldsymbol{\ell}$ and for all contraction vector $\boldsymbol{\rho}$ we have

(24) $$\dim_{\mathrm{P}} E_{\boldsymbol{\ell}}^{\boldsymbol{\rho}} < L + m.$$

The generated self-similar IFS



We fix the vector of slopes $\boldsymbol{\rho}$.

Lemma 3.1

There is a non-singular linear transformation F *which depends only on* $\boldsymbol{\rho}$ *such that*

$$F_{\boldsymbol{\rho}}(\mathfrak{b}, \boldsymbol{\tau}) = \boldsymbol{t}.$$

Theorem 3.2 (Hochman[4])

$$\dim_{\mathrm{P}} \left\{ \boldsymbol{t} \in \mathbb{R}^M : \mathcal{S}^{\boldsymbol{t}} \text{ does not satisfy the ESC} \right\} = M - 1. \tag{25}$$

[4] Michael Hochman. On self-similar sets with overlaps and inverse theorems for entropy in $\mathbb{R}^d$, 2015

Corollary 3.3

For a $\dim_{\mathrm{P}}$*-typical CPLIFS* $\mathcal{F}$*, the generated self-similar IFS* $\mathcal{S}_{\mathcal{F}}$ *satisfies the ESC.*

Theorem 3.4 (Raith, Simon, P.)

Let $\mathcal{F}$ *be a CPLIFS with generated self-similar system* $\mathcal{S}_{\mathcal{F}}$ *and attractor* Λ*. If* $\mathcal{S}_{\mathcal{F}}$ *satisfies the ESC, then*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}. \tag{26}$$

Theorem 3.5 (Raith, Simon, P.)

Fix a type $\boldsymbol{\ell}$ and a slope vector $\boldsymbol{\rho}$ with positive entries. For $\mathcal{L}_{m+L}$-almost every $(\mathfrak{b}, \boldsymbol{\tau}) \in \mathfrak{B}^{\boldsymbol{\ell}} \times \mathbb{R}^m$ we have

$$s^{\mathcal{F}} > 1 \implies \mathcal{L}_1(\Lambda^{(\mathfrak{b},\boldsymbol{\tau})}) > 0, \tag{27}$$

where $\Lambda^{(\mathfrak{b},\boldsymbol{\tau})}$ denotes the attractor of $\mathcal{F}^{(\boldsymbol{\rho},\mathfrak{b},\boldsymbol{\tau})}$.

Introduction

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Proof of Theorem 3.4

Limit-irreducibility

Proof of Theorem 3.4

We need to approximate the Markov diagram of the CPLIFS with finite subdiagrams.

Since $\mathbf{F}(s)$ is always irreducible, according to Seneta's results[5], it can be done if our CPLIFS has the following property.

[5] Eugene Seneta. *Non-negative matrices and Markov chains.* Springer Science & Business Media, 2006

We say that the CPLIFS $\mathcal{F}$ is limit-irreducible if there exists a $\mathcal{Y}$ finite refinement of $\mathcal{Z}_0$ such that for all $s \in (0, \dim_{\mathrm{H}} \Lambda]$ the matrix $\mathbf{F}(\mathcal{Y}, s)$ has right and left eigenvectors with nonnegative entries for the eigenvalue $\varrho(\mathbf{F}(\mathcal{Y}, s))$.

We call this finite partition $\mathcal{Y}$ a limit-irreducible partition and $(\mathcal{D}(\mathcal{Y}), \to)$ a limit-irreducible Markov diagram of $\mathcal{F}$. $\mathbf{F}(\mathcal{Y}, s)$ is the matrix associated to this diagram.

Proof of Theorem 3.4 cont.

Proposition 4.1

Let $\mathcal{F}$ be a limit-irreducible CPLIFS, and let $(\mathcal{D}, \to)$ be its limit-irreducible Markov diagram. For any $\varepsilon > 0$ there exists a $\mathcal{C} \subset \mathcal{D}$ finite subset such that

$$\varrho(\mathbf{F}(s)) - \varepsilon \leqslant \varrho(\mathbf{F}_{\mathcal{C}}(s)) \leqslant \varrho(\mathbf{F}(s)), \tag{28}$$

where $\mathbf{F}(s)$ is the matrix associated to $(\mathcal{D}, \to)$.

Proof of Theorem 3.4 cont.

As $\dim_{\mathrm{H}} \Lambda \leqslant s^{\mathcal{F}}$ always holds, we only need to prove the other direction.

Choose an arbitrary $t \in (0, s^{\mathcal{F}})$. By Lemma 2.1

$$0 < \Phi(t) < \log \varrho(\mathbf{F}(t)).$$

According to Proposition 4.1

$$\exists \mathcal{C} \subset \mathcal{D} \text{ finite} : 0 < \log \varrho(\mathbf{F}_{\mathcal{C}}(t)) = \Phi_{\mathcal{C}}(t).$$

Proof of Theorem 3.4 cont.

Theorem 4.2 (Simon, P.[6])

Let $\mathcal{F}$ be a self-similar graph directed IFS with attractor Λ and generated self-similar IFS $\mathcal{S}$. If $\mathcal{S}$ satisfies the ESC, then

$$\dim_{\mathrm{H}} \Lambda = \min\{1, s^{\mathcal{F}}\}.$$

It follows, that $\dim_{\mathrm{H}} \Lambda_{\mathcal{C}} = \min\{s_{\mathcal{C}}, 1\}$, where $s_{\mathcal{C}}$ is the unique root of $\Phi_{\mathcal{C}}(s)$.

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[6] R Dániel Prokaj and Károly Simon. Piecewise linear iterated function systems on the line of overlapping construction. *Nonlinearity*, 35(1):245, 2021

Proof of Theorem 3.4 cont.

$s^{\mathcal{F}} > 1$ implies $dim_{\mathrm{H}}\Lambda_{\mathcal{C}} = 1$, for a suitable finite and irreducible subdiagram $(\mathcal{C}, \rightarrow)$.

$s^{\mathcal{F}} \leqslant 1$ implies $s_{\mathcal{C}} \leqslant 1$ for all $\mathcal{C} \subset \mathcal{D}$.

$$0 < \Phi_{\mathcal{C}}(t) \implies t < s_{\mathcal{C}} = \dim_H \Lambda_{\mathcal{C}} \leqslant \dim_H \Lambda, \tag{29}$$

and it holds for any $t \in (0, s^{\mathcal{F}})$. Thus $s^{\mathcal{F}} \leqslant \dim_{\mathrm{H}} \Lambda$.

Altough limit-irreducibility is required in the proof, we do not need to assume that our CPLIFS have this property, as it is already granted by the ESC.

Lemma 5.1 (F. Hofbauer[7])

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS with Markov diagram $(\mathcal{D}, \to)$ and associated matrix $\mathbf{F}(s)$. If $\mathbf{F}(s)$ can be written in the form

$$\mathbf{F}(s) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

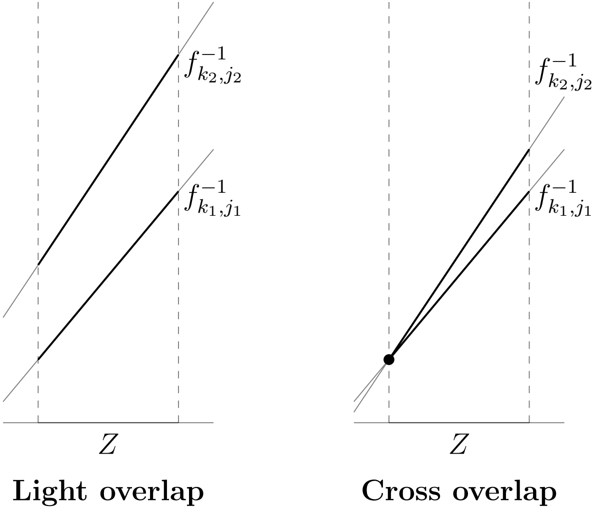
such that $\varrho(\mathbf{F}(s)) > \varrho(S)$, then $\mathcal{F}$ is limit-irreducible.

[7] Franz Hofbauer. Piecewise invertible dynamical systems. *Probability theory and related fields*, 72(3):359–386, 1986

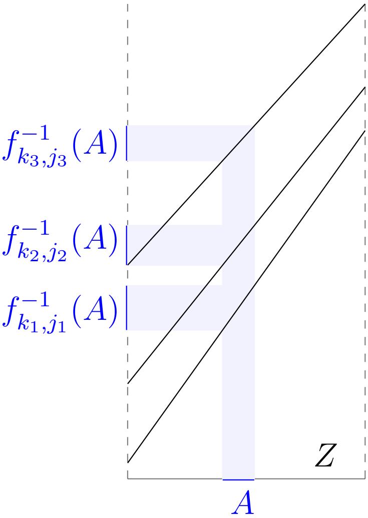
Lemma 5.1 always applies for systems without overlaps, where all the entries of $\mathbf{F}(s)$ are smaller than 1.

We have to investigate what happens in the overlapping cases, as multiple edges in $(\mathcal{D}, \rightarrow)$ might yield bigger than 1 entries in the associated matrix.

Two types of overlaps



Light overlaps



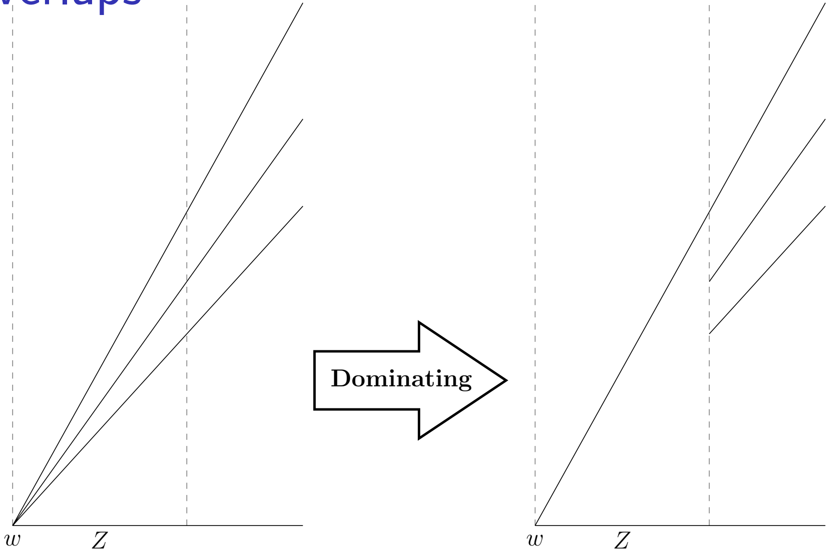
By choosing a finite refinement of $\mathcal{Z}_0$ that has sufficiently small entries, we can easily avoid having multiple edges in the diagram.

Cross overlaps

The case of cross overlaps is more complicated, as they induce nested sequences of intervals for any finite refinement of $\mathcal{Z}_0$.

The ESC implies that no crossing point can have a periodic orbit. Thus, $\varrho(S)$ won't grow too big if we use the branch with the largest expansion ratio among the crossing branches instead of the others.

Cross overlaps



Thank you for your attention!