Fractal dimensions of continuous piecewise linear iterated function systems

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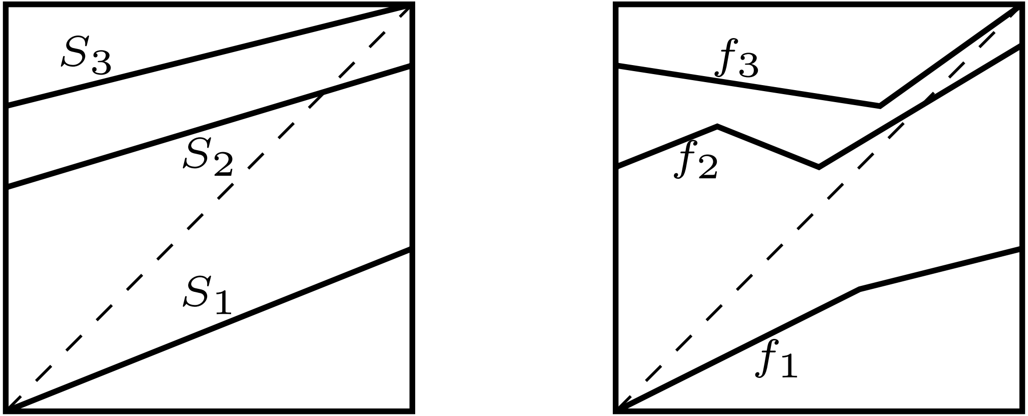


Self-similar IFS
$\mathcal{S} = \{S_1, S_2, S_3\}$

CPLIFS
$\mathcal{F} = \{f_1, f_2, f_3\}$

The Main Result

For packing dimension typical CPLIFS $\mathcal{F}$

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}. \tag{1}$$

The meaning of "packing dimension typical": the packing dimension of the parameters of the exceptional CPLIFS is less than the dimension of the parameter space.

An Iterated Function System (IFS) $\mathcal{F} = \{f_k\}_{k=1}^m$ on the line is a finite list of strict contractions on $\mathbb{R}$.

The attractor Λ of the IFS $\mathcal{F}$ is the unique non-empty compact set

(2)
$$\Lambda = \bigcup_{k=1}^{m} f_k(\Lambda).$$

By iterating formula (2), one obtains

(3)
$$\Lambda = \bigcup_{(i_1,\dots,i_n)\in[m]^n} f_{i_1\dots i_n}(\Lambda).$$

Here we used the common notation $f_{i_1\dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$.

Let I be the smallest non-empty compact interval such that $f_i(I) \subset I$ for all $i \in [m] := \{1, \ldots, m\}$.

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in[m]^n} I_{i_1\ldots i_n}, \tag{4}$$

where $I_{i_1\ldots i_n} := f_{i_1\ldots i_n}(I)$ are the `cylinder intervals`.
Thus these intervals form a natural cover of the attractor.

The natural dimension

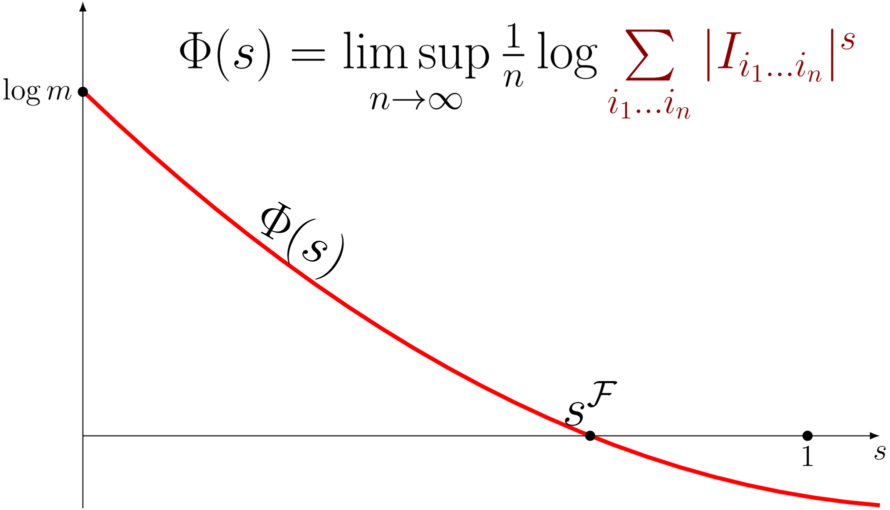
$$\Phi(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s. \tag{5}$$

It is easy to see that we can obtain $\Phi(s)$ above as a special case of the non-additive upper capacity topological pressure introduced by Barreira in [1, p. 5]. We call the unique zero of this function the natural dimension of $\mathcal{F}$.

$$s^{\mathcal{F}} := (\Phi)^{-1}(0). \tag{6}$$

The natural pressure function

$$\Phi(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\dots i_n} |I_{i_1\dots i_n}|^s$$



The Hausdorff dimension

The t-dimensional Hausdorff measure of the attractor is

$$(7) \quad \mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \Lambda \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| < \delta \right\} \right\},$$

where the infimum is taken over all $\{A_i\}$ covers.
The Hausdorff dimension of Λ is defined as

$$(8) \quad \dim_{\mathrm{H}} \Lambda = \inf\{t : \mathcal{H}^t(\Lambda) = 0\} = \sup\{t : \mathcal{H}^t(\Lambda) = \infty\}.$$

Barreira [1] also showed that

$$\dim_{\mathrm{H}} \Lambda \leqslant \overline{\dim}_{\mathrm{B}} \Lambda \leqslant \min\left\{1, s^{\mathcal{F}}\right\}. \tag{9}$$

Under what condition do we have equality?

Self-similar IFS

If our iterated function system is of the form

$$\mathcal{F} = \{f_k(x) = r_k \cdot x + t_k\}_{k=1}^{m}$$

then $\mathcal{F}$ is called self-similar. In this case

$$\Phi(s) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \ldots i_n} |r_{i_1} \cdots r_{i_n}|^s = \log \sum_{i=1}^{m} |r_i|^s \tag{10}$$

$$\Phi(s^{\mathcal{F}}) = 0 \iff \sum_{k=1}^{m} |r_k|^{s^{\mathcal{F}}} = 1. \tag{11}$$

Hence $s^{\mathcal{F}}$ is the similarity dimension.

The distance of two similarity mappings $g_1(x) = r_1 x + \tau_1$ and $g_2(x) = r_2 x + \tau_2$, $r_1, r_2 \in (-1,1) \setminus \{0\}$, on $\mathbb{R}$.

$$\text{dist}\,(g_1, g_2) := \begin{cases} |\tau_1 - \tau_2|, & \text{if } r_1 = r_2; \\ \infty, & \text{otherwise.} \end{cases} \tag{12}$$

Given a self-similar IFS $\mathcal{F} = \{f_k\}_{k=1}^M$ on $\mathbb{R}$. We say that $\mathcal{F}$ satisfies the Exponential Separation Condition (ESC) if there exists a $c > 0$ and a strictly increasing sequence of natural numbers $\{n_\ell\}_{\ell=1}^\infty$ such that

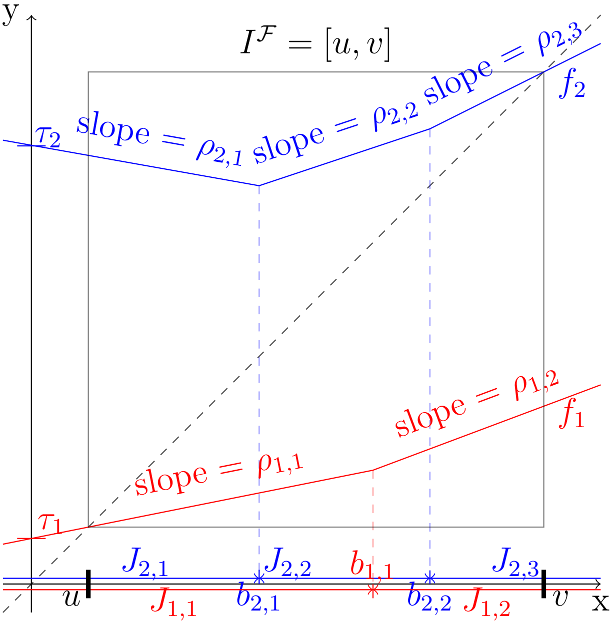
$$\text{dist}\,(f_{\bar{\imath}}, f_{\bar{\jmath}}) \geqslant c^{n_\ell} \text{ for all } \ell \text{ and for all } \bar{\imath}, \bar{\jmath} \in \{1, \ldots, M\}^{n_\ell},\ \bar{\imath} \neq \bar{\jmath}.$$

Self-similar IFS 2

Hochman [2] proved that for any self-similar IFS on the line that satisfies the ESC we have

$$\dim_{\mathrm{H}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}.$$

We managed to extend this result to CPLIFS, with the help of Markov diagrams.



$\mathcal{F} = \{f_k\}_{k=1}^m$, $\tau_k := f_k(0)$, $l(k)$ is the number of breaking points of f_k: $\{b_{k,1}, \ldots, b_{k,l(k)}\}$. $J_{k,i}$ are the intervals of linearity. $\boldsymbol{\ell} = (l(1), \ldots, l(m))$ is the type of $\mathcal{F}$ $L := l(1) + \cdots + l(m)$.

Fix a type $\boldsymbol{\ell} = (l(1), \dots, l(m))$ and a vector of contractions $\boldsymbol{\rho} \in ((-1,1)\backslash\{0\})^{L+m}$. We write $\mathfrak{b}, \boldsymbol{\tau}$ for the vector of breaking points and translations, respectively.

Let $\mathfrak{P}$ be a property that makes sense for every CPLIFS, and consider the exceptional set

$$E_{\boldsymbol{\ell}}^{\boldsymbol{\rho}} =: \left\{ (\mathfrak{b}, \boldsymbol{\tau}) \in \mathbb{R}^{L+m} : \mathcal{F}^{(\mathfrak{b},\boldsymbol{\tau},\boldsymbol{\rho})} \text{ does not have property } \mathfrak{P} \right\}. \tag{13}$$

We say that property $\mathfrak{P}$ holds $\dim_{\mathrm{P}}$-typically if for all type $\boldsymbol{\ell}$ and for all contraction vector $\boldsymbol{\rho}$ we have

$$\dim_{\mathrm{P}} E_{\boldsymbol{\ell}}^{\boldsymbol{\rho}} < L + m. \tag{14}$$

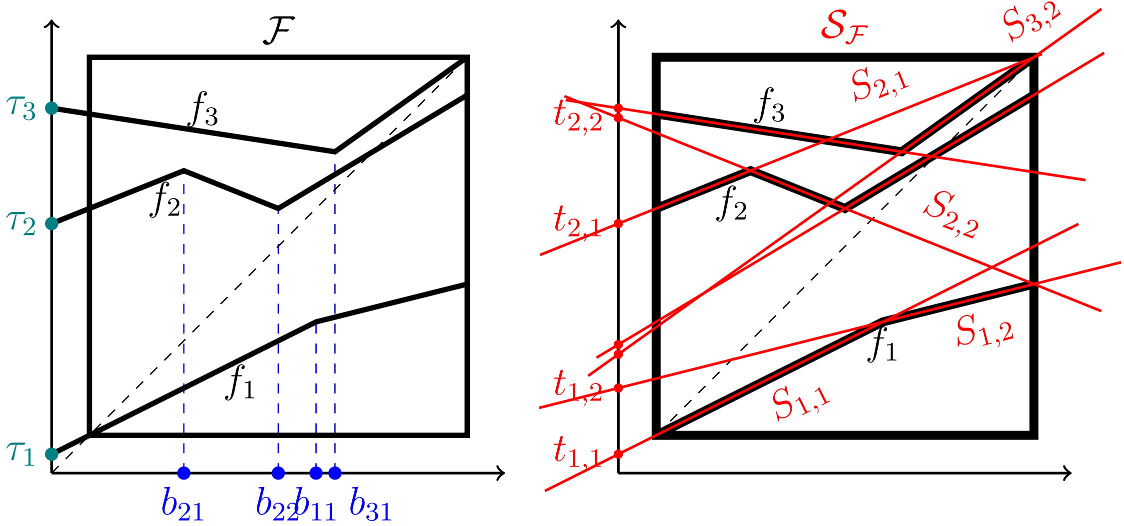
For a CPLIFS $\mathcal{F}$ the generated self-similar IFS

$$S_{\mathcal{F}} = \{S_{k,i}(x) = \rho_{k,i}x + t_{k,i}\}_{k \in [m], i \in [l(k)+1]}$$

consists of those similarity mappings on $\mathbb{R}$ whose graph coincide with the graph of f_k for some $k \in [m]$ on some interval of linearity $J_{k,i}$ of f_k.

$$S_{k,i}|_{J_{k,i}} = f_k|_{J_{k,i}}, \ \forall k \in [m], i \in [l(k)+1]$$

The generated self-similar IFS



[5, Fact 4.1] provides a connection between the translation parameters of a CPLIFS and its generated self-similar IFS.

Lemma 2.1

Let $\mathcal{F}$ be a CPLIFS, and let $\mathcal{S}_{\mathcal{F}}$ be the generated self-similar IFS. As earlier, let $\boldsymbol{t}$ be the translation vector of $\mathcal{S}_{\mathcal{F}}$, and write $\mathfrak{b}, \boldsymbol{\tau}$ for the vector of breaking points and translations, respectively.

There is a non-singular linear transformation F which depends only on $\boldsymbol{\rho}$ such that

$$F_{\boldsymbol{\rho}}(\mathfrak{b}, \boldsymbol{\tau}) = \boldsymbol{t}.$$

Fix a vector of contractions $\boldsymbol{r} := (r_1, \ldots, r_M) \in ((-1,1)\setminus\{0\})^M$. For a $\boldsymbol{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$ we consider the self-similar IFS associated to the vector of translations $\boldsymbol{t}$:

$$\mathcal{S}^{\boldsymbol{t}} := \{S_k(x) = r_k x + t_k\}_{k=1}^{M}.$$

Theorem 2.2 (Hochman [3, Theorem 1.10])

(15) $$\dim_{\mathrm{P}} \left\{\boldsymbol{t} \in \mathbb{R}^M : \mathcal{S}^{\boldsymbol{t}} \textit{ does not satisfy the ESC } \right\} = M - 1.$$

Lemma 2.1 and Theorem 2.2 together imply that our main result follows from the next theorem.

Theorem 2.3 (Simon, Raith, P.)

Let $\mathcal{F}$ be a CPLIFS with generated self-similar system $\mathcal{S}_{\mathcal{F}}$ and attractor Λ. If $\mathcal{S}_{\mathcal{F}}$ satisfies the ESC, then

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \min\left\{1, s^{\mathcal{F}}\right\}. \tag{16}$$

Let $I_k := f_k(I)$ and $\mathcal{I} = \cup_{k=1}^m I_k$. We define the expanding multi-valued mapping associated to $\mathcal{F}$ as

$$T : \mathcal{I} \mapsto \mathcal{P}(\mathcal{P}(I)), \qquad T(y) := \{\{x \in I : f_k(x) = y\}\}_{k=1}^m. \tag{17}$$

For $k \in [m], j \in [l(k) + 1]$, we define $f_{k,j} : J_{k,j} \mapsto I_k$ as the uniqe linear function that satisfies $f_k(x) = f_{k,j}(x), \forall x \in J_{k,j}$.

We call the expansive linear functions

$$\forall k \in [m], \forall j \in [l(k)+1] : f_{k,j}^{-1}$$

the branches of T.

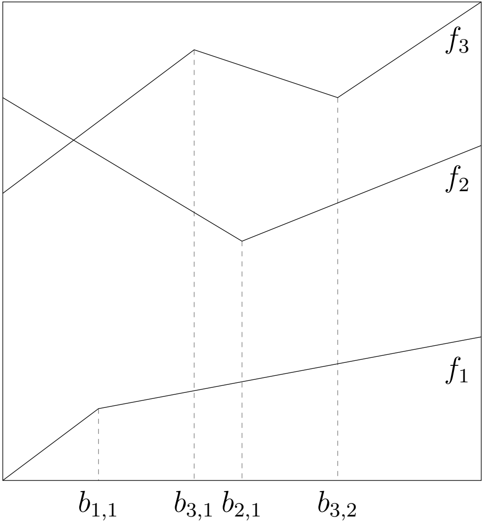
We define the set of critical points as

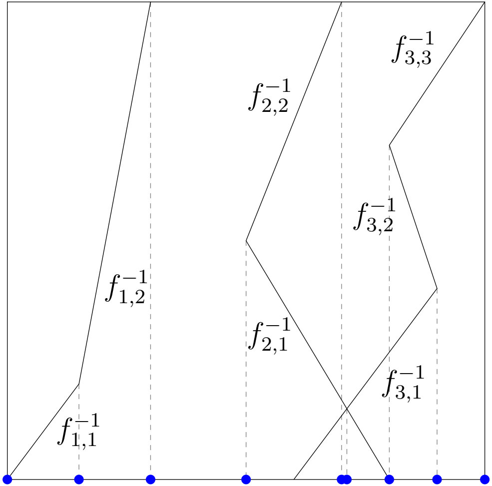
$$\mathcal{K} := \cup_{k=1}^{m} \{f_k(0), f_k(1)\} \bigcup \cup_{k=1}^{m} \cup_{j=1}^{l(k)} f_k(b_{k,j}) \bigcup$$
$$\left\{x \in \mathcal{I} \middle| \exists k_1, k_2 \in [m], \exists j_1 \in [l(k_1)], \exists j_2 \in [l(k_2)] : f_{k_1,j_1}^{-1}(x) = f_{k_2,j_2}^{-1}(x)\right\}.$$

We call the partition of $\mathcal{I}$ into closed intervals defined by the set of critical points $\mathcal{K}$ the monotonicity partition $\mathcal{Z}_0$ of $\mathcal{F}$. We call its elements monotonicity intervals.

That is, above monotonicity intervals T is always linear, and branches can only take the same value at the boundary.

The associated multi-valued mapping





Let $Z \in \mathcal{Z}_0$. We say that D is a successor of Z and we write $Z \to D$ if

$$\exists Z_0 \in \mathcal{Z}_0, Z' \in T(Z) : D = Z_0 \cap Z'. \tag{18}$$

Further, we write $Z \to_{k,j} D$ if

$$\exists Z_0 \in \mathcal{Z}_0 : D = Z_0 \cap f_{k,j}^{-1}(Z).$$

The set of successors of Z is denoted by $w(Z) := \{D | Z \to D\}$.

We say that $(\mathcal{D}, \to)$ is the Markov Diagram of $\mathcal{F}$ with respect to $\mathcal{Z}_0$ if $\mathcal{D}$ is the smallest set containing $\mathcal{Z}_0$ such that $\mathcal{D} = w(\mathcal{D})$.

We can similarly define the Markov diagram of $\mathcal{F}$ with respect to any finite partition $\mathcal{Z}_0'$ of $\mathcal{I}$.

One can imagine the Markov diagram as a (potentially infinitely big) directed graph, with vertex set $\mathcal{D}$.

Between $C, D \in \mathcal{D}$, we have a directed edge $C \to D$ if and only if $D \in w(C)$. We call the Markov diagram irreducible if there exists a directed path between any two intervals $C, D \in \mathcal{D}$.

Since the functions of a CPLIFS are always continuous on $\mathbb{R}$, we can always assume that $(\mathcal{D}, \to)$ is irreducible.

Associated matrix

We define the matrix $\mathbf{F}(s) := \mathbf{F}_{\mathcal{D}}(s)$ indexed by the elements of $\mathcal{D}$ as

$$(19) \qquad [\mathbf{F}(s)]_{C,D} := \begin{cases} \sum_{(k,j):C \to_{(k,j)} D} \left| f'_{k,j} \right|^s, \text{ if } C \to D \\ 0, \text{ otherwise.} \end{cases}$$

This matrix is often associated to self-similar graph directed iterated function systems. When the diagram is finite, our system is actually a GDIFS.

Let $\mathcal{C} \subset \mathcal{D}$. We write $\mathcal{E}_{\mathcal{C}}(n)$ for the set of n-length directed paths in the subgraph $(\mathcal{C}, \rightarrow)$.

Assume that $(\mathcal{C}, \rightarrow)$ is irreducible. Each path in $(\mathcal{C}, \rightarrow)$ of infinite length represents a point in the invariant set $\Lambda_{\mathcal{C}} \subset \Lambda$. We define the natural pressure of these sets as

$$\Phi_{\mathcal{C}}(s) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\mathbf{k}} |I_{\mathbf{k}}|^{s}, \tag{20}$$

where the sum is taken over all $\mathbf{k} = (k_1, \ldots k_n)$ for which $\exists j_1, \ldots j_n : ((k_1, j_1), \ldots, (k_n, j_n)) \in \mathcal{E}_{\mathcal{C}}(n)$.

As an operator, $(\mathbf{F}_{\mathcal{D}}(s))^n$ is always bounded in the l^∞-norm. Thus we can define

$$\varrho(\mathbf{F}_{\mathcal{C}}(s)) := \lim_{n\to\infty} \|(\mathbf{F}_{\mathcal{C}}(s))^n\|_\infty^{1/n}.$$

Lemma 3.1

Let $\mathcal{C} \subset \mathcal{D}$. If $(\mathcal{C}, \to)$ is irreducible, then

(21) $$\Phi_{\mathcal{C}}(s) \leqslant \log \varrho(\mathbf{F}_{\mathcal{C}}(s)).$$

If $(\mathcal{C}, \to)$ is irreducible and finite, then

(22) $$\Phi_{\mathcal{C}}(s) = \log \varrho(\mathbf{F}_{\mathcal{C}}(s)).$$

Proof of Theorem 2.3

To prove Theorem 2.3, we need to approximate the original Markov diagram with finite subdiagrams.

Since $\mathbf{F}(s)$ is always irreducible, according to Seneta's results [6, Theorem 1], it can be done if our CPLIFS has the following property.

We say that the CPLIFS $\mathcal{F}$ is limit-irreducible if there exists a $\mathcal{Y}$ finite refinement of $\mathcal{Z}_0$ such that for all $s \in (0, \dim_{\mathrm{H}} \Lambda]$ the matrix $\mathbf{F}(\mathcal{Y}, s)$ has right and left eigenvectors with nonnegative entries for the eigenvalue $\varrho(\mathbf{F}(\mathcal{Y}, s))$.

We call this finite partition $\mathcal{Y}$ a limit-irreducible partition and $(\mathcal{D}(\mathcal{Y}), \to)$ a limit-irreducible Markov diagram of $\mathcal{F}$. $\mathbf{F}(\mathcal{Y}, s)$ is the matrix associated to this diagram.

Proof of Theorem 2.3 cont.

Proposition 4.1

Let $\mathcal{F}$ be a limit-irreducible CPLIFS, and let $(\mathcal{D}, \to)$ be its limit-irreducible Markov diagram. For any $\varepsilon > 0$ there exists a $\mathcal{C} \subset \mathcal{D}$ finite subset such that

$$\varrho(\mathbf{F}(s)) - \varepsilon \leqslant \varrho(\mathbf{F}_{\mathcal{C}}(s)) \leqslant \varrho(\mathbf{F}(s)), \tag{23}$$

where $\mathbf{F}(s)$ is the matrix associated to $(\mathcal{D}, \to)$.

Proof of Theorem 2.3 cont.

Choose an arbitrary $t \in (0, s_{\mathcal{F}})$. By Lemma 3.1

$$0 < \Phi(t) < \log \varrho(\mathbf{F}(t)).$$

According to Proposition 4.1

$$\exists \mathcal{C} \subset \mathcal{D} \text{ finite} : 0 < \log \varrho(\mathbf{F}_{\mathcal{C}}(t)) = \Phi_{\mathcal{C}}(t).$$

Proof of Theorem 2.3 cont.

Theorem 4.2 (Simon, P. [5, Corollary 7.2])

Let $\mathcal{F}$ be a self-similar graph directed IFS with attractor Λ and generated self-similar IFS $\mathcal{S}$. If $\mathcal{S}$ satisfies the ESC, then

$$\dim_{\mathrm{H}} \Lambda = \min\{1, s^{\mathcal{F}}\}.$$

It follows, that $\dim_{\mathrm{H}} \Lambda_{\mathcal{C}} = \min\{s_{\mathcal{C}}, 1\}$, where $s_{\mathcal{C}}$ is the unique root of $\Phi_{\mathcal{C}}(s)$.

Proof of Theorem 2.3 cont.

$s^{\mathcal{F}} > 1$ implies $dim_{\mathrm{H}}\Lambda_{\mathcal{C}} = 1$, for a suitable finite and irreducible subdiagram $(\mathcal{C}, \rightarrow)$.

$s^{\mathcal{F}} \leqslant 1$ implies $s_{\mathcal{C}} \leqslant 1$ for all $\mathcal{C} \subset \mathcal{D}$.

(24) $$0 < \Phi_{\mathcal{C}}(t) \implies t < s_{\mathcal{C}} = \dim_H \Lambda_{\mathcal{C}} \leqslant \dim_H \Lambda,$$

and it holds for any $t \in (0, s^{\mathcal{F}})$. Thus $s^{\mathcal{F}} \leqslant \dim_{\mathrm{H}} \Lambda$.

Altough limit-irreducibility is required in the proof, we do not need to assume that our CPLIFS have this property, as it is already granted by the ESC.

Lemma 5.1 (F. Hofbauer [4, Corollary 1/ii])

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a CPLIFS with Markov diagram $(\mathcal{D}, \to)$ and associated matrix $\mathbf{F}(s)$. If $\mathbf{F}(s)$ can be written in the form

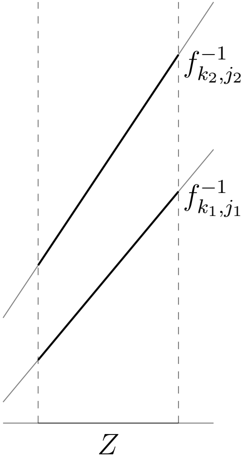
$$\mathbf{F}(s) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

such that $\varrho(\mathbf{F}(s)) > \varrho(S)$, then $\mathcal{F}$ is limit-irreducible. Here P, Q, R, S are appropriate dimensional block matrices.

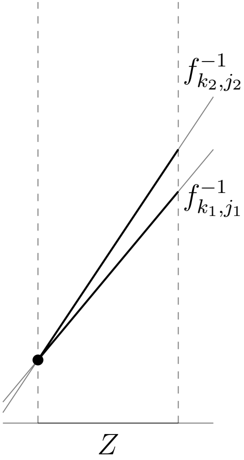
Lemma 5.1 always applies for systems without overlaps, where all the entries of $\mathbf{F}(s)$ are smaller than 1.

We have to investigate what happens in the overlapping cases, as multiple edges in $(\mathcal{D}, \rightarrow)$ might yield bigger than 1 entries in the associated matrix.

Two types of overlaps

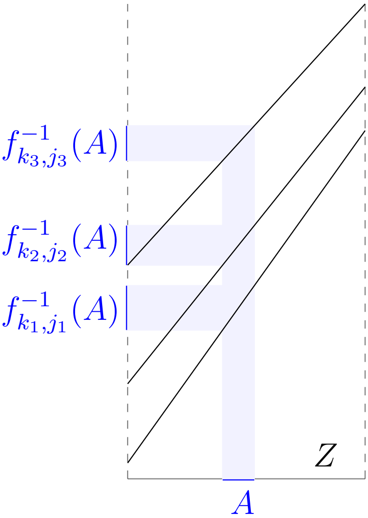


Light overlap



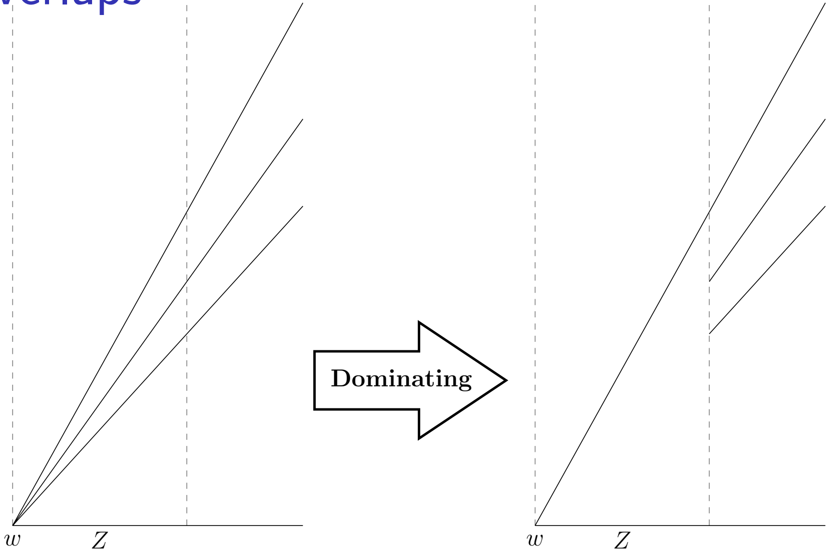
Cross overlap

Light overlaps



By choosing a finite refinement of $\mathcal{Z}_0$ that has sufficiently small entries, we can easily avoid having multiple edges in the diagram.

Cross overlaps



Thank you for your attention!

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